RAMANUJAN'S THEORY OF THETA FUNCTIONS

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Ramanujan to Hardy 12 January 1920

I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary ϑ-functions.

Definitions

Definition

Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

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$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

$$\begin{aligned}
\varphi(q) :=& f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \\
\psi(q) :=& f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \\
f(-q) :=& f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.
\end{aligned}$$

Jacobi Triple Product Identity

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

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Definition

Complete Elliptic Integral of the First Kind

$$K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

where k, $0 < k < 1$, is the *modulus*.

Modular Equation

Definition

Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, ℓ, and $\ell' := \sqrt{1-\ell^2}$, respectively, where $0 < k, \ell < 1$. Suppose that, for $n \in \mathbb{Z}^+$,

$$n\frac{K'}{K} = \frac{L'}{L}. \qquad (1)$$

A relation between k and ℓ induced by (1) is called a **modular equation of degree** n. Set

$$\alpha = k^2 \qquad \text{and} \qquad \beta = \ell^2.$$

We often say that β has degree n over α.

Main Theorem in Elliptic Functions

$$q = \exp(-\pi K'/K),$$

$$(a)_0 := 1, \quad (a)_n := a(a+1)\cdots(a+n-1), \quad n \geq 1.$$

$${}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \qquad |x| < 1.$$

$$\begin{aligned}\varphi^2(q) &= {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) \\ &= \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} =: \frac{2}{\pi} K(k).\end{aligned}$$

Multiplier

$$m = \frac{\varphi^2(q)}{\varphi^2(q^n)} = \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \beta)}.$$

Examples of Modular Equations

$$z = \varphi^2(q), \qquad z_n = \varphi^2(q^n)$$

Third Order

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1,$$

Seventh Order

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1.$$

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1	17	3,9	3,13,39	5,27,135	11,13,143
3	19	5,25	3,21,63	7,9,63	11,21,231
7	23	3,5,15	3,29,87	7,17,119	13,19,247
11	31	3,7,21	5,7,35	7,25,175	15,17,255
13	47	3,9,27	5,11,55	9,15,135	
15	51	3,11,33	5,19,95	9,23,207	

Catalogue

Theorem

Let $x = \alpha = k^2$.

(i) $$\varphi(q) = \sqrt{z},$$

(ii) $$\varphi(-q) = \sqrt{z}(1-x)^{1/4},$$

(iii) $$\varphi(-q^2) = \sqrt{z}(1-x)^{1/8},$$

(iv) $$\varphi(q^2) = \sqrt{z}\sqrt{\tfrac{1}{2}\left(1+\sqrt{1-x}\right)},$$

(v) $$\varphi(q^4) = \tfrac{1}{2}\sqrt{z}\left(1+(1-x)^{1/4}\right),$$

(vi) $$\varphi(\sqrt{q}) = \sqrt{z}\left(1+\sqrt{x}\right)^{1/2},$$

(vii) $$\varphi(-\sqrt{q}) = \sqrt{z}\left(1-\sqrt{x}\right)^{1/2}.$$

Catalogue

Theorem

$$
\begin{aligned}
&\text{(i)} & \psi(q) &= \sqrt{\tfrac{1}{2}z}(x/q)^{1/8}, \\
&\text{(ii)} & \psi(-q) &= \sqrt{\tfrac{1}{2}z}\left(x(1-x)/q\right)^{1/8}, \\
&\text{(iii)} & \psi(q^2) &= \tfrac{1}{2}\sqrt{z}(x/q)^{1/4}, \\
&\text{(iv)} & \psi(q^4) &= \tfrac{1}{2}\sqrt{\tfrac{1}{2}z}\left\{\left(1-\sqrt{1-x}\right)/q\right\}^{1/2}, \\
&\text{(v)} & \psi(q^8) &= \tfrac{1}{4}\sqrt{z}\{1-(1-x)^{1/4}\}/q, \\
&\text{(vi)} & \psi(\sqrt{q}) &= \sqrt{z}\left\{\tfrac{1}{2}(1+\sqrt{x})\right\}^{1/4}(x/q)^{1/16}, \\
&\text{(vii)} & \psi(-\sqrt{q}) &= \sqrt{z}\left\{\tfrac{1}{2}(1-\sqrt{x})\right\}^{1/4}(x/q)^{1/16}.
\end{aligned}
$$

Catalogue

Theorem

$$
\begin{aligned}
&\text{(i)} & f(q) &= \sqrt{z}2^{-1/6}\left\{x(1-x)/q\right\}^{1/24}, \\
&\text{(ii)} & f(-q) &= \sqrt{z}2^{-1/6}(1-x)^{1/6}(x/q)^{1/24}, \\
&\text{(iii)} & f(-q^2) &= \sqrt{z}2^{-1/3}\left\{x(1-x)/q\right\}^{1/12}, \\
&\text{(iv)} & f(-q^4) &= \sqrt{z}4^{-1/3}(1-x)^{1/24}(x/q)^{1/6}.
\end{aligned}
$$

Examples of multipliers

Entry (p. 351)

If β and the multiplier m have degree 3, then

$$m^2 = \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{1-\beta}{1-\alpha}} - \sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}.$$

Entry (p. 351)

If β and the multiplier m have degree 5, then

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}.$$

Examples of multipliers

Entry (p. 351)

If β and the multiplier m have degree 7, then

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}.$$

Entry (p. 351)

If β and the multiplier m have degree 9, then

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}.$$

Examples of multipliers

Entry (p. 352)

If β and the multiplier m have degree 13, then

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}.$$

Examples of multipliers

Entry (p. 352)

If β and the multiplier m have degree 17, then

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}\left\{1 + \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8}\right\}.$$

More General Transformation Formulas

Recall that m can be represented as a quotient of hypergeometric functions. Thus, the formulas for m can be regarded as transformation formulas for hypergeometric functions. Can any of these transformation formulas be generalized by replacing the parameters $\frac{1}{2}$, $\frac{1}{2}$, 1 by functions of α and β?

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$$
{}_2F\left(a, b; 2b; \frac{4x}{(1+x)^2}\right) = (1+x)^{2a}\,{}_2F_1\left(a, a+\frac{1}{2}-b; b+\frac{1}{2}; x^2\right)
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$$

How did Ramanujan derive formulas for m?
How did Ramanujan derive modular equations, in general.

Sums of Theta Functions

Theorem (Entry 31, Chapter 16)

Let, for each positive integer n,

$$U_n = a^{n(n+1)/2} b^{n(n-1)/2},$$
$$V_n = a^{n(n-1)/2} b^{n(n+1)/2}.$$

Then

$$f(a,b) = \sum_{r=0}^{n-1} U_r \, f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right).$$

Sums of Theta Functions

$$T(x,q) := \sum_{n=-\infty}^{\infty} x^n q^{n^2}, \quad x \neq 0, \quad |q| < 1$$

Theorem (Schröter's Formula)

For positive integers a, b,

$$T(x,q^a)T(y,q^b) = \sum_{n=0}^{a+b-1} y^n q^{bn^2} T(xyq^{2bn}, q^{a+b}) \times T(x^{-b}y^a q^{2abn}, q^{ab^2+a^2b}).$$

Example

If μ is an odd positive integer,

$$\begin{aligned}
&\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) \\
&= q^{\mu^2/4-\mu/4}\psi(q^{2\mu(\mu^2-\nu^2)})f(q^{\mu+\mu\nu}, q^{\mu-\mu\nu}) \\
&\quad + \sum_{m=0}^{(\mu-3)/2} q^{\mu m(m+1)} f\left(q^{(\mu+2m+1)(\mu^2-\nu^2)},\right. \\
&\quad \left. q^{(\mu-2m-1)(\mu^2-\nu^2)}\right) f(q^{\mu+\nu+2\nu m}, q^{\mu-\nu-2\nu m}).
\end{aligned}$$

Eta Function Identities

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1$$

$$f(-q) = (q;q)_\infty = e^{-2\pi i\tau/24}\eta(\tau),\ \ q = e^{2\pi i\tau}.$$

Theorem

Let

$$P = \frac{f(-q)}{q^{1/4}f(-q^7)} \quad \textit{and} \quad Q = \frac{f(-q^3)}{q^{3/4}f(-q^{21})}.$$

Then

$$PQ + \frac{7}{PQ} = \left(\frac{Q}{P}\right)^2 - 3 + \left(\frac{P}{Q}\right)^2.$$

Theorem

Let S be the set consisting of one copy of the positive integers and one additional copy of those positive integers that are multiples of 7. If $k \in \mathbb{Z}^+$, the number of partitions of $2k$ into even elements of S is equal to the number of partitions of $2k+1$ into odd elements of S.

Theorem

Let S denote the set consisting of two copies, say in colors orange and blue, of the positive integers and one additional copy, say in color red, of those positive integers that are not multiples of 3. Let $A(N)$ and $B(N)$ denote the number of partitions of $2N$ into odd elements and even elements, respectively, of S. Then, for $N \geq 1$,

$$A(N) = B(N).$$

Combinatorics of Theta Function Identities

$$\left(\frac{\alpha^3}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} = 1.$$

$$(-q;q^2)_\infty^2(-q,-q^5;q^6)_\infty + (q;q^2)_\infty^2(q,q^5;q^6)_\infty$$
$$= 2(-q^2;q^2)_\infty^2(-q^2,-q^4;q^6)_\infty.$$

$$(a_1,a_2,\ldots,a_n;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty$$

$A(3) = 12 = B(3)$, with the twelve representations in odd and even elements being given respectively by

$$\begin{aligned} 5_o + 1_o &= 5_o + 1_b = 5_o + 1_r = 5_b + 1_o = 5_b + 1_b \\ &= 5_b + 1_r = 5_r + 1_o = 5_r + 1_b \\ &= 5_r + 1_r = 3_o + 3_b = 3_o + 1_o + 1_b + 1_r \\ &= 3_b + 1_o + 1_b + 1_r, \end{aligned}$$

$$\begin{aligned} 6_o = 6_b &= 4_o + 2_o = 4_o + 2_b = 4_o + 2_r \\ &= 4_b + 2_o = 4_b + 2_b = 4_b + 2_r \\ &= 4_r + 2_o = 4_r + 2_b = 4_r + 2_r \\ &= 2_o + 2_b + 2_r. \end{aligned}$$

Sums of Squares and Triangular Numbers

Definition

$r_k(n)$ denotes the number of representations of n as the sum of k squares.
$t_k(n)$ denotes the number of representations of n as the sum of k triangular numbers.

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$$\varphi^8(q) = 1 + 16\sum_{n=1}^{\infty} \frac{n^3 q^n}{1-(-q)^n}.$$

$$r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3 \quad \text{(Jacobi)}$$

Sums of Squares and Triangular Numbers

$\sigma(n)$ denotes the sum of the positive divisors of n.

$$q\psi^4(q^2) = \sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{4n+2}}.$$

$$t_4(n) = \sigma(2n+1) \quad \text{(Legendre)}$$

Example $n = 4$

$$\sigma(9) = 1 + 3 + 9 = 13$$

$$1 + 0 + 3 + 0, \quad 4 \cdot 3 = 12, \quad 1 + 1 + 1 + 1$$

Sums of Numbers of the Form $m^2 + mn + n^2$

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}$$

If $\chi_0(n)$ denotes the principal character modulo 3, then

$$a^2(q) = 1 + 12 \sum_{n=1}^{\infty} \chi_0(n) \frac{nq^n}{1-q^n}.$$

If $\left(\frac{n}{3}\right)$ denotes the Legendre symbol, then

$$a^3(q) = 1 - 9 \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{n^2 q^n}{1-q^n} + 27 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1+q^n+q^{2n}}.$$

Arithmetic Interpretation

$r_{k,Q}(n) = \#$ of representations of the positive integer n as a sum of k positive definite quadratic forms
$Q = Q(a,b,c) = ax^2 + bxy + cy^2$.
Let $Q = x^2 + xy + y^2$. Then

$$\sum_{N=1}^{\infty} r_{2,Q}(N) q^N = 12 \sum_{N=1}^{\infty} \sigma(N;\chi_0) q^N,$$

where $\sigma(N;\chi_0)$ denotes the sum of the divisors of N that are not multiples of 3. Thus, for $N \geq 1$,

$$r_{2,Q}(N) = 12\sigma(N;\chi_0).$$

Example

Example. Let $N = 3$. Then

$$\begin{aligned}
3 &= (\pm 1)^2 + (\pm 1)(\pm 1) + (\pm 1)^2 + 0 \\
&\quad (m = n = 1; m = n = -1) \quad 4 \\
&= (-2)^2 + (-2)(1) + 1^2 + 0 \\
&\quad (m = 1, n = -2; m = -1; n = 2; \\
&\quad m = 2, n = -1; m = -2, n = 1) \quad 8
\end{aligned}$$

Class Invariants and Singular Moduli

$$\chi(q) := (-q; q^2)_\infty$$

If n is any positive rational number and $q = \exp(-\pi\sqrt{n})$,

$$G_n := 2^{-1/4} q^{-1/24} \chi(q),$$
$$g_n := 2^{-1/4} q^{-1/24} \chi(-q).$$

$$\alpha_n := \alpha(e^{-\pi\sqrt{n}})$$

is the *singular modulus*.

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24},$$
$$g_n = 2^{-1/12}(1-\alpha_n)^{1/12}\alpha_n^{-1/24}$$

Examples of Class Invariants

$$G_5 = \left(\frac{1+\sqrt{5}}{2}\right)^{1/4},$$

$$G_9 = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3},$$

$$G_{13} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4},$$

$$G_{69} = \left(\frac{5+\sqrt{23}}{\sqrt{2}}\right)^{1/12} \left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{1/8} \times \left(\sqrt{\frac{6+3\sqrt{3}}{4}} + \sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{1/2}$$

Examples of Class Invariants

$$G_{117} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4} \left(2\sqrt{3}+\sqrt{13}\right)^{1/6} \times \left(3^{1/4}+\sqrt{4+\sqrt{3}}\right)$$

Examples of Class Invariants

$$G_{1353} = (3+\sqrt{11})^{1/4}(5+3\sqrt{3})^{1/4}$$
$$\times \left(\frac{11+\sqrt{123}}{2}\right)^{1/4}\left(\frac{6817+321\sqrt{451}}{4}\right)^{1/12}$$
$$\times \left(\sqrt{\frac{17+3\sqrt{33}}{8}}+\sqrt{\frac{25+3\sqrt{33}}{8}}\right)^{1/2}$$
$$\times \left(\sqrt{\frac{561+99\sqrt{33}}{8}}+\sqrt{\frac{569+99\sqrt{33}}{8}}\right)^{1/2}$$

Approximations to π

$$\pi \approx \frac{24}{\sqrt{142}} \log\left(\frac{\sqrt{10+11\sqrt{2}}+\sqrt{10+7\sqrt{2}}}{2}\right),$$

$$\pi \approx \frac{12}{\sqrt{190}} \log\left((3+\sqrt{10})(2\sqrt{2}+\sqrt{10})\right).$$

approximate π to 15, 18 decimals, resp.

Examples of Singular Moduli

$$k_{210} = (\sqrt{2}-1)^4(2-\sqrt{3})^2(\sqrt{7}-\sqrt{6})^4 \\ \times (8-3\sqrt{7})^2(\sqrt{10}-3)^4(4-\sqrt{15})^4 \\ \times (\sqrt{15}-\sqrt{14})^2(6-\sqrt{35})^2.$$

$$\alpha_3 = \frac{2-\sqrt{3}}{4},$$

$$\alpha_6 = (2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2,$$

$$\alpha_{13} = \frac{1}{2}\left(\frac{\sqrt{13}-3}{2}\right)^3, \\ \times \left(\sqrt{\frac{7+\sqrt{13}}{4}} - \sqrt{\frac{3+\sqrt{13}}{4}}\right)^4$$

Examples of Singular Moduli

$$\alpha_{15} = \frac{1}{16}\left(\frac{\sqrt{5}-1}{2}\right)^4 (2-\sqrt{3})^2(4-\sqrt{15}),$$

$$\alpha_{55} = 4\left(\sqrt{5}-2\right)^2 (10-3\sqrt{11})(3\sqrt{5}-2\sqrt{11})$$
$$\times\left(\sqrt{\frac{7+\sqrt{5}}{8}}-\sqrt{\frac{\sqrt{5}-1}{8}}\right)^{12},$$
$$\times\left(\sqrt{\frac{4+\sqrt{5}}{2}}-\sqrt{\frac{2+\sqrt{5}}{2}}\right)^{4}$$

$$\alpha_{58} = (13\sqrt{58}-99)^2(99-70\sqrt{2})^2,$$

Examples of Singular Moduli

$$\alpha_{190} = \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^4 (37\sqrt{19}-51\sqrt{10})^2$$
$$\times (2\sqrt{5}-\sqrt{19})^4(\sqrt{19}-3\sqrt{2})^4.$$

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$$a_{m,n} := ne^{-(\pi/4)(n-1)\sqrt{m/n}} \times \frac{\psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{m/n}})\varphi^2(-e^{-2\pi\sqrt{m/n}})}$$

$$a_{3,7} = 2 - \sqrt{3},$$

$$a_{3,13} = \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13}-3}{8}}\right)^8,$$

The Rogers–Ramanujan Continued Fraction

$$R(q) := \frac{q^{1/5}}{1} \; + \; \frac{q}{1} \; + \; \frac{q^2}{1} \; + \; \frac{q^3}{1} \; + \cdots, \qquad |q| < 1$$

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}$$

Theorem

Let $a = 60^{1/4}$, $b = 2 - \sqrt{3} + \sqrt{5}$. *If*

$$2c = \frac{a+b}{a-b}\sqrt{5} + 1,$$

then

$$R(e^{-6\pi}) = \sqrt{c^2+1} - c.$$

G. H. Hardy to Ramanujan 26 March 1913

What I should like above all is a definite proof of some of your results concerning continued fractions of the type

$$\frac{x}{1} + \frac{x^2}{1} + \frac{x^3}{1} + \cdots ;$$

and I am quite sure that the wisest thing you can do, in your own interests, is to let me have one as soon as possible.

How to Spend Christmas Eve

G. H. Hardy to Ramanujan 24 December 1913

If you will send me your proof written out carefully (so that it is easy to follow), I will (assuming that I agree with it–of which I have very little doubt) try to get it published for you in England. Write it in the form of a paper 'On the continued fraction

$$\frac{x}{1} + \frac{x^2}{1} + \frac{x^3}{1} + \cdots,$$

giving a full proof of the principal and most remarkable theorem, viz. that the fraction can be expressed in finite terms when $x = e^{-\pi\sqrt{n}}$, when $\underline{n}$ is rational.

The Rogers–Ramanujan Functions

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}.$$

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L. J. Rogers
G. N. Watson
David Bressoud
A. J. F. Biagioli
Hamza Yesilyurt

Identities for $G(q)$ and $H(q)$

$$G(q^{11})H(q) - q^2 G(q)H(q^{11}) = 1.$$

$$G(-q^6)H(-q) - qH(-q^6)G(-q) = \frac{\chi(q^2)\chi(q^3)}{\chi(q)\chi(q^6)}.$$

$$\chi(q) := (-q; q^2)_\infty$$

$$\frac{G(q)G(q^{54}) + q^{11}H(q)H(q^{54})}{G(q^{27})H(q^2) - q^5 G(q^2)H(q^{27})} = \frac{\chi(-q^3)\chi(-q^{27})}{\chi(-q)\chi(-q^9)}.$$

Identities for $G(q)$ and $H(q)$

$$
\begin{aligned}
&\left\{G(q^2)G(q^{23}) + q^5 H(q^2)H(q^{23})\right\} \\
&\quad \times \left\{G(q^{46})H(q) - q^9 G(q)H(q^{46})\right\} \\
&= \chi(-q)\chi(-q^{23}) + q + \frac{2q^2}{\chi(-q)\chi(-q^{23})}.
\end{aligned}
$$

$$
\begin{aligned}
&\left\{G(q)G(q^{94}) + q^{19} H(q)H(q^{94})\right\} \\
&\quad \times \left\{G(q^{47})H(q^2) - q^9 G(q^2)H(q^{47})\right\} \\
&= \chi(-q)\chi(-q^{47}) + 2q^2 + \frac{2q^4}{\chi(-q)\chi(-q^{47})} \\
&\quad + q\sqrt{4\chi(-q)\chi(-q^{47}) + 9q^2 + \frac{8q^4}{\chi(-q)\chi(-q^{47})}}.
\end{aligned}
$$

Eisenstein Series

$$P(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}, \tag{2}$$

$$Q(q) := 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k}, \tag{3}$$

$$R(q) := 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1-q^k}. \tag{4}$$

Eisenstein Series

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$$E_4(\tau) = Q(q), \quad E_6(\tau) = R(q), \quad q = e^{2\pi i \tau}.$$

Eisenstein Series

$$Q(q) = z^4(1 + 14x + x^2),$$
$$Q(q^2) = z^4(1 - x + x^2),$$
$$R(q) = z^6(1 + x)(1 - 34x + x^2),$$
$$R(q^2) = z^6(1 + x)(1 - \tfrac{1}{2})(1 - 2x).$$

Eisenstein Series

$$
\begin{aligned}
Q(q) &= z^4(1+14x+x^2),\\
Q(q^2) &= z^4(1-x+x^2),\\
R(q) &= z^6(1+x)(1-34x+x^2),\\
R(q^2) &= z^6(1+x)(1-\tfrac{1}{2})(1-2x).
\end{aligned}
$$

$$
\begin{aligned}
Q(q) = \frac{f^{10}(-q)}{f^2(-q^5)} &+ 250qf^4(-q)f^4(-q^5)\\
&+ 3125q^2\frac{f^{10}(-q^5)}{f^2(-q)}
\end{aligned}
$$

Eisenstein Series

$$T_{2k}(q) := 1 + \sum_{n=1}^{\infty} (-1)^n \left\{ (6n-1)^{2k} q^{n(3n-1)/2} \right.$$
$$\left. + (6n+1)^{2k} q^{n(3n+1)/2} \right\},$$

Eisenstein Series

$$T_{2k}(q) := 1 + \sum_{n=1}^{\infty} (-1)^n \left\{ (6n-1)^{2k} q^{n(3n-1)/2} + (6n+1)^{2k} q^{n(3n+1)/2} \right\},$$

$$\frac{T_2(q)}{(q;q)_\infty} = P,$$

$$\frac{T_4(q)}{(q;q)_\infty} = 3P^2 - 2Q,$$

$$\frac{T_6(q)}{(q;q)_\infty} = 15P^3 - 30PQ + 16R.$$

Eisenstein Series

Corollary

For $n \geq 1$, let $\sigma(n) = \sum_{d|n} d$, and define $\sigma(0) = -\frac{1}{24}$. Let n denote a nonnegative integer. Then

$$-24 \sum_{\substack{j+k(3k\pm1)/2=n \\ j,k\geq 0}} (-1)^k \sigma(j) = \begin{cases} (-1)^r(6r-1)^2, & \text{if } n = r(3r-1)/2, \\ (-1)^r(6r+1)^2, & \text{if } n = r(3r+1)/2, \\ 0, & \text{otherwise.} \end{cases}$$

Companion Corollary

$$\sigma(n) = \sum_{k=-\infty}^{\infty} (-1)^{k+1} \frac{k(3k+1)}{2} p\left(n - \frac{k(3k+1)}{2}\right).$$

Modular Equations and Approximations to π

$$f_n(q) := nP(q^{2n}) - P(q^2),$$

Twelve values of n, namely, $n = 2, 3, 4, 5, 7, 11, 15, 17, 19, 23, 31$, and 35.
$n = 2$ and 4 are in Chapter 17 in Ramanujan's second notebook; the remaining ten values and for $n = 9$ and $n = 25$ are in Chapter 21 of his second notebook.

$$f_7(q) = 3z(q)z(q^7)\left(1 + \sqrt{\alpha(q)\alpha(q^7)} + \sqrt{(1-\alpha(q))(1-\alpha(q^7))}\right).$$

Modular Equations and Approximations to π

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} (6n+1) \frac{(\frac{1}{2})_n^3}{n!^3} \frac{1}{4^n},$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} (42n+5) \frac{(\frac{1}{2})_n^3}{n!^3} \frac{1}{2^{6n}},$$

$$\frac{5\sqrt{5}}{\pi} = \sum_{k=0}^{\infty} (28k+3) \frac{\left(\frac{1}{6}\right)_k \left(\frac{1}{2}\right)_k \left(\frac{5}{6}\right)_k}{k!^3} \left(\frac{3}{5}\right)^{3k}$$

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Ramanujan's Theories of Elliptic Functions to Alternative Bases

$$q = \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right).$$

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Ramanujan's Theories of Elliptic Functions to Alternative Bases

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$r = 2$, classical theory,
$r = 3$, cubic theory,
$r = 4$, quartic theory,
$r = 6$, sextic theory.

Functions to Alternative Bases

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2},$$

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

Theorem (Cubic Transformation)

For $|x|$ sufficiently small,

$$
{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\left(\frac{1-x}{1+2x}\right)^3\right) = (1+2x)\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};x^3\right).
$$

Functions to Alternative Bases

Theorem

Modular Equation of Degree 5 in the Cubic Theory. *If* β *has degree 5, then*

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} + 3\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1.$$

Theorem

Modular Equation of Degree 5 in the Cubic Theory. *If β has degree 5, then*

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Theorem

Modular Equation of Degree 5 in the Quartic Theory. *If β has degree 5, then*

$$\begin{aligned}(\alpha\beta)^{1/2} &+ \{(1-\alpha)(1-\beta)\}^{1/2} \\ &+ 8\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} \\ &\times \left((\alpha\beta)^{1/6} + \{(1-\alpha)(1-\beta)\}^{1/6}\right) = 1.\end{aligned}$$

Ramanujan's Cubic Class Invariant

$$\lambda_n = \frac{1}{3\sqrt{3}} \frac{f^6(q)}{\sqrt{q} f^6(q^3)}$$
$$= \frac{1}{3\sqrt{3}} \left(\frac{\eta\left(\frac{1+i\sqrt{n/3}}{2}\right)}{\eta\left(\frac{1+i\sqrt{3n}}{2}\right)} \right)^6,$$

where $q = e^{-\pi\sqrt{n/3}}$.

$$\lambda_9 = 3,$$
$$\lambda_{17} = 4 + \sqrt{17},$$
$$\lambda_{73} = \left(\sqrt{\frac{11+\sqrt{73}}{8}} + \sqrt{\frac{3+\sqrt{73}}{8}} \right)^6.$$

Entry

For each positive integer n and $|ab| < 1$,

$$\sum_{-\frac{n}{2}<r\leq\frac{n}{2}} \left(\sum_{\substack{k=-\infty \\ k\equiv r \pmod{n}}}^{\infty} (a^{1/n})^{k(k+1)/2}(b^{1/n})^{k(k-1)/2} \right)^n = f(a,b)F_n(ab),$$

where

$$F_n(x) := 1 + 2nx^{(n-1)/2} + \cdots, \qquad n \geq 3.$$

Identities in Two Variables

$$F_2(x) = \varphi(\sqrt{x}),$$

$$F_3(x) = \left(\frac{f^9(-x)}{f^3(-x^3)} + 27x\frac{f^9(-x^3)}{f^3(-x)}\right)^{1/3}$$

$$= \frac{\psi^3(x)}{\psi(x^3)} + 3x\frac{\psi^3(x^3)}{\psi(x)}.$$

Entry (page 321, Ramanujan's Second Notebook)

If $\left(\frac{n}{3}\right)$ *denotes the Legendre symbol, then for* $|ab| < 1$,

$$f^3(a^2b, ab^2) + af^3(b, a^3b^2) + bf^3(a, a^2b^3) = f(a,b)\left(1 + 6\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{a^nb^n}{1-a^nb^n}\right).$$

Identities in Two Variables

Entry (page 328, Ramanujan's Second Notebook)

If $|ab| < 1$,

$$\left\{f(a,b) - f(a^6b^3, a^3b^6)\right\}^3 = \frac{f(a^3,b^3)}{f(a^6b^3,a^3b^6)} f^3(a^2b, ab^2) - f^3(a^6b^3, a^3b^6).$$

Identities in Two Variables

Entry (p. 207, Lost Notebook)

If

$$P = \frac{f(-\lambda^{10}q^7, -\lambda^{15}q^8) + \lambda q f(-\lambda^5 q^2, -\lambda^{20}q^{13})}{q^{1/5} f(-\lambda^{10}q^5, -\lambda^{15}q^{10})},$$

$$Q = \frac{\lambda f(-\lambda^5 q^4, -\lambda^{20}q^{11}) - \lambda^3 q f(-q, -\lambda^{25}q^{14})}{q^{-1/5} f(-\lambda^{10}q^5, -\lambda^{15}q^{10})},$$

then

$$P - Q = 1 + \frac{f(-q^{1/5}, -\lambda q^{2/5})}{q^{1/5} f(-\lambda^{10}q^5, -\lambda^{15}q^{10})},$$

$$PQ = 1 - \frac{f(-\lambda, -\lambda^4 q^3) f(-\lambda^2 q, -\lambda^3 q^2)}{f^2(-\lambda^{10}q^5, -\lambda^{15}q^{10})},$$

Identities in Two Variables

$$P^5 - Q^5 = 1 + 5PQ + 5P^2Q^2 + \frac{f(-q, -\lambda^5 q^2) f^5(-\lambda^2 q, -\lambda^3 q^2)}{q\, f^6(-\lambda^{10} q^5, -\lambda^{15} q^{10})}.$$

Identities in Two Variables

Let $\lambda = 1$.

$$P = \frac{f(-q^7, -q^8) + qf(-q^2, -q^{13})}{q^{1/5} f(-q^5)} = \frac{1}{R(q)},$$

$$Q = \frac{f(-q^4, -q^{11}) - qf(-q, -q^{14})}{q^{-1/5} f(-q^5)} = R(q).$$

Identities in Two Variables

Let $\lambda = 1$.

$$P = \frac{f(-q^7, -q^8) + qf(-q^2, -q^{13})}{q^{1/5} f(-q^5)} = \frac{1}{R(q)},$$

$$Q = \frac{f(-q^4, -q^{11}) - qf(-q, -q^{14})}{q^{-1/5} f(-q^5)} = R(q).$$

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)},$$

$$PQ = 1,$$

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}.$$

Further Appearances of Theta Functions in Ramanujan's Lost Notebook

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)(n+2)/2}}{(q;q)_n(1-q^{2n+1})} = qf(q,q^7),$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n(1-q^{2n+1})} = f(q^3,q^5),$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)(n+2)/2}}{(q;q)_n(q;q^2)_{n+1}} = \frac{qf(-q,-q^7)}{\varphi(-q)},$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)/2}}{(q;q)_n(q;q^2)_{n+1}} = \frac{f(-q^3,-q^5)}{\varphi(-q)}.$$

False Theta Functions

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}$$

$$\sum_{n=0}^{\infty} q^{3n^2+2n}(1-q^{2n+1})$$

$$\sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1})$$

Epilogue

Ramanujan to Hardy 12 January 1920

I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary ϑ-functions.