Eisenstein Series for subgroups of $SL(2, \mathbf{Z})$

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Eisenstein series on the full modular group

Define, for $q = e^{2\pi i\tau}$, $\operatorname{Im}\tau > 0$,

$$P(\tau) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \quad Q(\tau) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},$$

$$R(\tau) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}.$$

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- The series Q and R are the unique normalized Eisenstein series of weight 4 and 6 for $SL(2,\mathbb{Z})$.
- The weighted algebra of all integral weight holomorphic modular forms for $SL(2,\mathbb{Z})$ is generated by Q and R.
- P is a quasimodular form on $SL(2,\mathbb{Z})$, satisfying

$$P\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2P(\tau)+sc(c\tau+d), \quad \forall \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

where $s \in \mathbb{C}$ is the coefficient of affinity of P.

In 1914, Ramanujan proved that

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3}, \qquad q\frac{dR}{dq} = \frac{PR - Q^2}{2}.$$

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He derives, in an elementary way, a classical differential equation satisfied by the Weierstrass $\wp$-function and a new identity involving the square of the Weierstrass ζ-function.

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Oddly, this important result from the theory of modular forms

- does not utilize the theory of modular forms, and
- does not employ complex analysis or the notion of double periodicity.

Ramanujan proved the differential equations by equating coefficients in the trigonometric series identities

$$\left(\frac{1}{4}\cot\frac{1}{2}\theta+\sum_{k=1}^{\infty}\frac{q^k\sin(k\theta)}{1-q^k}\right)^2$$
$$=\left(\frac{1}{4}\cot\frac{1}{2}\theta\right)^2+\sum_{k=1}^{\infty}\frac{q^k\cos(k\theta)}{(1-q^k)^2}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}(1-\cos(k\theta))$$

and

$$\left(\frac{1}{8}\cot^2\frac{1}{2}\theta+\frac{1}{12}+\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}(1-\cos(k\theta))\right)^2$$
$$=\left(\frac{1}{8}\cot^2\frac{1}{2}\theta+\frac{1}{12}\right)^2+\frac{1}{12}\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}(5+\cos(k\theta)).$$

Ramanujan's differential equations for Eisenstein series play a role in proving many of the results in his notebooks, including the lost notebook.

In 2007, H. showed that the differential equations imply the parametric representations

$$P(q) = z^2(1-5x) + 12x(1-x)z\frac{dz}{dx}, \tag{1}$$

$$Q(q) = z^4(1+14x+x^2), \tag{2}$$

$$R(q) = z^6(1+x)(1-34x+x^2), \tag{3}$$

where

$$z = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right).$$

These parameterizations and the preceding differential equations are main ingredients in proofs of many of Ramanujan's modular equations.

Ramanujan likely thought about how to extend the differential equations for Eisenstein series to generalizations.

On page 332 of the Lost notebook, Ramanujan writes

$$``\frac{1^r}{e^{1^s x}-1}+\frac{2^r}{e^{2^s x}-1}+\frac{3^r}{e^{3^s x}-1}+\cdots,$$

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What Ramanujan meant by the above entry is not clear.

The series does not fit into the theory of elliptic functions or the theory of modular forms, except when $s=1$.

This entry, and others, have inspired a number of generalizations of classical results.

- In the 1970s, V. Ramamani, a student of Prof. Venkatacheliengar, derived analogous coupled system of differential equations for modular forms on $\Gamma_0(2)$ and $\Gamma^0(2)$.
- In 2007, T. H. extended Ramamani's method to derive a similar coupled set of differential equations for modular forms on the theta subgroup.
- In 2008, R. Maier, using the theory of modular forms, has shown that a similar set of differential equations are satisfied by Eisenstein series on $\Gamma_0(3)$ and $\Gamma_0(4)$.

Can Ramanujan's methods be used to derive differential equations for modular forms on subgroups of $SL(2,\mathbb{Z})$?

Are there interesting consequences of these differential equations, or results analogous to classical identities?

Theorem

For $\alpha \neq 1/2$*, define*

$$e_\alpha(q) = 1 + \frac{4}{\cot(\pi\alpha)} \sum_{n=1}^{\infty} \frac{\sin(2n\pi\alpha) q^n}{1-q^n},$$

$$P_\alpha(q) = 1 - \frac{8}{\csc^2(\pi\alpha)} \sum_{n=1}^{\infty} \frac{\cos(2n\pi\alpha) n q^n}{1-q^n},$$

$$Q_\alpha(q) = 1 - \frac{8}{\cot(\pi\alpha)\csc^2(\pi\alpha)} \sum_{n=1}^{\infty} \frac{\sin(2n\pi\alpha) n^2 q^n}{1-q^n}.$$

Then

$$q\frac{d}{dq} e_\alpha = \frac{\csc^2(\pi\alpha)}{4}\left(e_\alpha P_\alpha - Q_\alpha\right)$$

$$q\frac{d}{dq} P_\alpha = \frac{\csc^2(\pi\alpha)}{4} P_\alpha^2 - \frac{1}{2}\cot^2(\pi\alpha) e_\alpha Q_\alpha + \frac{1}{2}\cot(\pi\alpha)\cot(2\pi\alpha) e_{1-2\alpha} Q_\alpha$$

$$q\frac{d}{dq} Q_\alpha = \frac{1}{4} Q_\alpha P_\alpha \csc^2(\pi\alpha) + \frac{1}{2} P_{1-2\alpha} Q_\alpha \csc^2(2\pi\alpha) - \frac{1}{2} e_{1-2\alpha}^2 Q_\alpha \cot^2(2\pi\alpha)$$
$$+ \frac{3}{2} e_\alpha e_{1-2\alpha} Q_\alpha \cot(\pi\alpha)\cot(2\pi\alpha) - e_\alpha^2 Q_\alpha \cot^2(\pi\alpha).$$

The theorem is proven by

- Equating coefficients in Ramanujan's first trigonometric series identity after replacing the variable θ by $\theta + \pi\alpha$,
- Using the fact that the sum of the residues of an elliptic function on a period parallelogram are zero, and that $f(z,x,q) =$

$$\frac{\theta_1(z+x+\pi\alpha \mid q)\theta_1(z-x+\pi\alpha \mid q)\theta_1(z+\pi(1-2\alpha) \mid q)}{\theta_1^3(z \mid q)}$$

is an elliptic function in z with period π and $\pi\tau$, $q = e^{2\pi i\tau}$, where $\theta_1(z \mid q) = 2q^{1/8}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sin(2n+1)z$.

- Noting that $f(z)$ has only a pole of order 3 in a period parallelogram, so

$$0 = Res(f;0) = \lim_{z\to 0} \frac{1}{2}\frac{\partial^2}{\partial z^2}(z^3 f(z))$$
$$= \frac{z^3 f(z)}{2}\left[\left(\frac{\partial}{\partial z}\log z^3 f(z)\right)^2 + \frac{\partial^2}{\partial z^2}\log z^3 f(z)\right].$$

- Deducing that

$$\left[\left(\frac{\partial}{\partial z}\log z^3 f(z)\right)^2 + \frac{\partial^2}{\partial z^2}\left(\log z^3 f(z)\right)\right]_{z=0} = 0,$$

  and equating coefficients of x in this identity.
- Equating coefficients in Ramanujan's second trigonometric series identity after replacing θ by $\theta + \pi\alpha$.

Let $\chi : \{0, 1, \ldots, N-1\} \to \mathbb{Z}$, and denote

$$\sigma_k(n \mid \chi) = \sum_{d|n} \chi(d \mod N) d^k.$$

Let $\chi_3(n)$ denote the Jacobi symbol modulo -3. Then

$$e_{1/3}(q) = 1+6\sum_{n=1}^{\infty} \sigma_0(n \mid \chi_3) q^n, \quad P_{1/3}(q) = 1+3\sum_{n=1}^{\infty} \sigma_1(n; -2, 1, 1) q^n,$$

$$Q_{1/3}(q) = 1-9\sum_{n=1}^{\infty} \sigma_2(n \mid \chi_3) q^n.$$

When $\alpha = 1/3$, we recover the following recent result of Maier.

Theorem

$$q\frac{d}{dq}e_{1/3} = \frac{e_{1/3}P_{1/3} - Q_{1/3}}{3}, \qquad q\frac{d}{dq}P_{1/3} = \frac{P_{1/3}^2 - e_{1/3}Q_{1/3}}{3},$$

$$q\frac{d}{dq}Q_{1/3} = P_{1/3}Q_{1/3} - e_{1/3}^2 Q_{1/3}.$$

- The series $e_{1/3}$ and $Q_{1/3}$ are normalized Eisenstein series on $\Gamma_0(3)$, and $P_{1/3}(q)$ is quasimodular on $\Gamma_0(3)$.
- Further changes of variables in the aforementioned trigonometric series identities produce differential equations involving Eisenstein series on conjugate subgroups of $\Gamma_0(3)$ in $SL(2,\mathbb{Z})$.
- The series $e_{1/3}(q)$ and $Q_{1/3}(q)$ are equal to the Borwein's cubic null theta functions, $a(q,0)$ and $b^3(q,0)$, respectively, where

$$a(q,z) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} z^{m-n},$$

$$b(q,z) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \omega^{m-n} z^n, \quad \omega = \exp(2\pi i/3),$$

while $P_{1/3}(q)$ is the logarithmic derivative of $c(q,0)$,

$$c(q,z) = \sum_{m,n=-\infty}^{\infty} q^{\left(m+\frac{1}{3}\right)^2+\left(m+\frac{1}{3}\right)\left(n+\frac{1}{3}\right)+\left(n+\frac{1}{3}\right)^2} z^{m-n}.$$

The Borwein's cubic theta function identity $a^3(q) + b^3(q) = c^3(q)$ is a direct consequence of these differential equations.

Corollary

$$e_{1/3}^3 - Q_{1/3} = 27q\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3}.$$

Proof.

The result follows by logarithmically differentiating each side of the claimed identity, and by applying the elementary relation

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} = \frac{(q^3;q^3)_\infty^2}{(q;q^3)_\infty(q^2;q^3)_\infty}.$$

□

Let $'$ denote the operator $q\frac{d}{dq}$. In 1956, R. Rankin gave a general method for deriving identities satisfied by modular forms and their derivatives. For example, with $\Delta = \frac{Q^3-R^2}{1728}$, we have

$$4QQ'' - 5(Q')^2 = 960\Delta, \qquad 6RR'' - 7(R')^2 = -3024Q\Delta.$$

Corollary

$$\begin{aligned} 0 &= e''_{1/3} - \frac{2}{3}P_{1/3}e'_{1/3} + \frac{2}{9}Q'_{1/3} \\ &= Q''_{1/3} + e^2_{1/3}Q'_{1/3} - 4Q_{1/3}P'_{1/3} + 5e_{1/3}Q_{1/3}e'_{1/3} \\ &= Q_{1/3}Q''_{1/3} - (Q'_{1/3})^2 - Q^2_{1/3}P'_{1/3} + 2e_{1/3}Q^2_{1/3}e'_{1/3}, \end{aligned}$$

and

$$e_{1/3}e''_{1/3} - 2(e'_{1/3})^2 = 6qQ_{1/3}\frac{(q^3;q^3)^9_\infty}{(q;q)^3_\infty},$$

$$Q_{1/3}Q''_{1/3} - \frac{4}{3}(Q'_{1/3})^2 = -9qe_{1/3}Q^2_{1/3}\frac{(q^3;q^3)^9_\infty}{(q;q)^3_\infty},$$

$$P''_{1/3} - \frac{2}{3}P_{1/3}P'_{1/3} + \frac{4}{3}Q_{1/3}e'_{1/3} = 9qQ_{1/3}\frac{(q^3;q^3)^9_\infty}{(q;q)^3_\infty}.$$

Theorem (D. Zagier, 1994)

Let

$$D = \frac{P^2 - Q}{12}\frac{\partial}{\partial P} + \frac{PQ - R}{3}\frac{\partial}{\partial Q} + \frac{PR - Q^2}{2}\frac{\partial}{\partial R}$$

and define

$$[f, g]_{D,n} = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s}\binom{n+\ell-1}{r} D^r f D^s g.$$

Then the subalgebra generated by Q and R is closed under the bracket operator $[\,]_n = [\,]_{D,n}$.

For example, with

$$1728\Delta = Q^3 - R^2,$$

we may derive

$$[Q, R]_1 = -3456\Delta, \quad [Q, \Delta]_1 = 4R\Delta, \quad [R, \Delta]_1 = 6Q^2\Delta,$$

$$[Q, Q]_2 = 4800\Delta, \quad [Q, R]_2 = 0, \quad [R, R]_2 = -21168Q\Delta,$$

$$[\Delta, \Delta]_2 = -13Q\Delta^2.$$

With

$$\Delta_{1/3} = e_{1/3}^3 - Q_{1/3},$$

and

$$D_{1/3} = \frac{e_{1/3}P_{1/3} - Q_{1/3}}{3}\frac{\partial}{\partial e_{1/3}} + \frac{P_{1/3}^2 - e_{1/3}Q_{1/3}}{3}\frac{\partial}{\partial P_{1/3}} + (P_{1/3}Q_{1/3} - e_{1/3}^2 Q_{1/3})\frac{\partial}{\partial Q_{1/3}}$$

we may similarly derive

$$[e_{1/3}, Q_{1/3}]_1 = -Q_{1/3}\Delta_{1/3}, \quad [e_{1/3}, \Delta_{1/3}]_1 = -Q_{1/3}\Delta_{1/3},$$
$$[Q_{1/3}, \Delta_{1/3}]_1 = 3e_{1/3}^2 Q_{1/3}\Delta_{1/3}, \quad [e_{1/3}, e_{1/3}]_2 = \frac{4}{9}Q_{1/3}\Delta_{1/3},$$
$$[e_{1/3}, Q_{1/3}]_2 = e_{1/3}^2 Q_{1/3}\Delta_{1/3}, \quad [Q_{1/3}, Q_{1/3}]_2 = -4e_{1/3}Q_{1/3}^2\Delta_{1/3},$$
$$[\Delta_{1/3}, \Delta_{1/3}]_2 = -4e_{1/3}Q_{1/3}\Delta_{1/3}^2.$$

Theorem

Let

$$D_{1/3} = \frac{e_{1/3}P_{1/3} - Q_{1/3}}{3}\frac{\partial}{\partial e_{1/3}} + \frac{P_{1/3}^2 - e_{1/3}Q_{1/3}}{3}\frac{\partial}{\partial P_{1/3}} + (P_{1/3}Q_{1/3} - e_{1/3}^2 Q_{1/3})\frac{\partial}{\partial Q_{1/3}}$$

and define

$$[f,g]_{D,n} = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s}\binom{n+\ell-1}{r} D^r f D^s g.$$

Then the subalgebra generated by $e_{1/3}$ *and* $Q_{1/3}$ *is closed under the bracket operator* $[\,]_n = [\,]_{D,n}$.

We may also imitate a number of identities in Ramanujan's Lost Notebook using cubic Eisenstein series instead of those on the full modular group:

Corollary

For $0 < q < 1$, $q = e^{2\pi i t}$,

$$\int_0^q \sqrt{e_{1/3}(t) Q_{1/3}(t)} \frac{dt}{t} = \log \left(\frac{e_{1/3}^{3/2}(q) - Q_{1/3}^{1/2}(q)}{e_{1/3}^{3/2}(q) + Q_{1/3}^{1/2}(q)} \right).$$

Corollary

For $\alpha_{\ell,mn} \in \mathbb{Q}$,

$$\sum_{m,n=-\infty}^{\infty} \left(m^2 + n^2 + mn + m + n + \frac{1}{3} \right)^k q^{\left(m+\frac{1}{3}\right)^2 + \left(n+\frac{1}{3}\right)\left(m+\frac{1}{3}\right) + \left(n+\frac{1}{3}\right)^2}$$

$$= \sum_{\ell+m+n=k+1} \alpha_{\ell,m,n} P_{1/3}^{\ell} e_{1/3}^{m} Q_{1/3}^{n}.$$

Setting $r = 4$, we obtain a set of differential equations equivalent to those for Eisenstein series on $\Gamma_0(4)$:

$$q\frac{d}{dq}e_{1/4} = \frac{e_{1/4}P_{1/4} - Q_{1/4}}{2}, \quad q\frac{d}{dq}P_{1/4} = \frac{P_{1/4}^2 - e_{1/4}Q_{1/4}}{2},$$

$$q\frac{d}{dq}Q_{1/4} = Q_{1/4}\frac{P_{1/2} + P_{1/4} - 2e_{1/4}^2}{2}.$$

We may show that

$$2P_{1/4}(q) - P_{1/2}(q) = Q_{1/4}(q)/e_{1/4}(q) = \frac{1}{3}(8P(q^4) - 6P(q^2) + P(q)).$$

Setting $Q_{1/4}/e_{1/4} := B^2$, we obtain a R. Maier's coupled system:

$$q\frac{d}{dq}(e_{1/4}^2) = e_{1/4}^2P_{1/4} - e^2B^2,$$

$$2q\frac{d}{dq}P_{1/4} = P_{1/4}^2 - e_{1/4}^2B^2,$$

$$q\frac{d}{dq}(B^2) = P_{1/4}^2B^2 - e_{1/4}^2B^2.$$

When $\alpha = \frac{1}{6}$, we derive

$$q\frac{d}{dq}e_{1/6} = e_{1/6}P_{1/6} - Q_{1/6},$$

$$q\frac{d}{dq}P_{1/6} = \frac{2P_{1/6}^2 - 3Q_{1/6}e_{1/6} + Q_{1/6}e_{2/3}}{2},$$

$$q\frac{d}{dq}Q_{1/6} = Q_{1/6}\frac{4P_{2/3} + 6P_{1/6} - 18e_{1/6}^2 + 9e_{1/6}e_{2/3} - e_{2/3}^2}{6}.$$

Note that

$$e_{1-2\alpha} = 2 - e_{2\alpha}, \quad Q_{1-2\alpha} = 2 - Q_{2\alpha}, \quad \text{and} \quad P_{1-2\alpha} = P_{2\alpha}. \quad (4)$$

Thus, cubic theta functions and their logarithmic derivatives appear above in the form of $e_{2/3}$ and $P_{2/3}$.

Corollary

Let $\alpha = \frac{5+\sqrt{5}}{10}$*. Then*

$$e_{1/5}e''_{1/5} - (e'_{1/5})^2 - \alpha Q_{1/5}e'_{1/5} - \alpha e^2_{1/5}P'_{1/5} + \alpha e_{1/5}Q'_{1/5} = 0.$$

$$e_{1/6}e''_{1/6} - (e'_{1/6})^2 - Q_{1/6}e'_{1/6} - e^2_{1/6}P'_{1/6} + e_{1/6}Q'_{1/6} = 0.$$

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More generally, we have

Corollary

Let $\beta = \frac{\csc^2(\pi/r)}{4}$*. Then*

$$e_{1/r}e''_{1/r} - (e'_{1/r})^2 - \beta Q_{1/r}e'_{1/r} - \beta e^2_{1/r}P'_{1/r} + \beta e_{1/r}Q'_{1/r} = 0.$$

If we denote $d/d\tau$ by a dot, and define

$$u_4 = \dot{u} - u^2 \qquad \text{and} \qquad u_{k+2} = \dot{u}_k - k u u_k,$$

then the quasimodular form $P(\tau)$ satisfies a third order differential equation

$$0 = u_8 + 24u_4^2$$

called the Chazy equation.

R. Maier shows that the series $u = (2\pi i/r)P_{1/r}$ satisfy the "generalized Chazy equations"

$$0 = u_4 u_8 - u_6^2 + 8u_4^3, \qquad \text{when } r = 4,$$
$$0 = u_4 u_8^2 - u_6^2 u_8 + 24u_4^3 u_8 - 15u_4^2 u_6^2 + 144u_4^5, \qquad \text{when } r = 3.$$

Is there a generalized Chazy equation satisfied by $P_{1/r}$ for every $r \geqslant 4$?

The coupled system of differential equations for $\Gamma_0(2)$ and conjugate subgroups derived by Ramamani and H. cannot be derived from the preceeding differential equations.

They result from Ramanujan's trigonometric series identities after variable changes corresponding to quasi-periods of the classical Jacobi theta functions.

We also need an identity of Ramamani involving the cube of the Weierstrass zeta function

$$\left(\frac{1}{4}\cot\frac{1}{2}\theta+\sum_{k=1}^{\infty}\frac{q^k\sin(k\theta)}{1-q^k}\right)^3=\left(\frac{1}{4}\cot\frac{1}{2}\theta\right)^3-\frac{3}{2}\sum_{k=1}^{\infty}\frac{q^k\sin(k\theta)}{(1-q^k)^3}$$

$$+\frac{3}{4}\sum_{k=1}^{\infty}\frac{(k+1)q^k\sin(k\theta)}{(1-q^k)^2}-\frac{1}{16}\sum_{k=1}^{\infty}\frac{(2k^2+1)q^k\sin(k\theta)}{1-q^k}$$

$$+\frac{3}{8}\cot\frac{\theta}{2}\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}+\frac{3}{2}\left(\sum_{k=1}^{\infty}\frac{q^k\sin(k\theta)}{1-q^k}\right)\left(\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}\right).$$

For $j = 2, 3$, define

$$e_1(q) = \frac{1}{6} + 4\sum_{n=1}^{\infty} \frac{nq^n}{1+q^n},$$

$$e_j(q) = -\frac{1}{12} + (-1)^{j+1} \sum_{\substack{n\geqslant 1 \\ n \text{ odd}}}^{\infty} \frac{nq^{n/2}}{1+(-1)^j q^{n/2}},$$

$$\mathcal{P}_1(q) = 1 + 8\sum_{n=1}^{\infty} \frac{(-1)^{n-1} nq^n}{1-q^n}, \quad \mathcal{P}_j(q) = -8\sum_{n=1}^{\infty} \frac{(-1)^{jn} nq^{n/2}}{1-q^n},$$

$$\mathcal{Q}_1(q) = 1 - 16\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^3 q^n}{1-q^n}, \quad \mathcal{Q}_j(q) = 16\sum_{n=1}^{\infty} \frac{(-1)^{jn} n^3 q^{n/2}}{1-q^n}.$$

Then for $j \in \{1, 2, 3\}$,

$$q\frac{d\mathcal{P}_j}{dq} = \frac{\mathcal{P}_j^2 - \mathcal{Q}_j}{4}, \quad q\frac{d\mathcal{Q}_j}{dq} = \mathcal{P}_j\mathcal{Q}_j - 6e_j\mathcal{Q}_j, \quad q\frac{de_j}{dq} = \frac{\mathcal{P}_j e_j}{2} - \frac{\mathcal{Q}_j}{12}.$$

As in the cubic case, we obtain results analogous to classical results and those appearing in Ramanujan's Lost Notebook.

Theorem

If

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{2}, \qquad \psi^2(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

then

$$36e_1^2 - \mathcal{Q}_1 = 64q\psi^8(q), \qquad 36e_2^2 - \mathcal{Q}_2 = \varphi^8(-q^{1/2})/4,$$

$$36e_3^2 - \mathcal{Q}_3 = \frac{\psi^{16}(q^{1/2})}{4\psi^8(q)} = \varphi^8(q^{1/2})/4.$$

Corollary

Let $0 < q < 1$*. For* $j = 1, 2, 3$*,*

$$\log\left(\frac{6e_j(q) - \mathcal{Q}_j^{1/2}(q)}{6e_j(q) + \mathcal{Q}_j^{1/2}(q)}\right) = \int_0^q \sqrt{\mathcal{Q}_j(t)}\,\frac{dt}{t}.$$

Corollary

Let

$$\mathcal{T}_{2k}(q) = 1 + \sum_{n=1}^{\infty} (2n+1)^{2k} q^{n(n+1)/2},$$

$$\mathcal{U}_{2k}(q) = 2^{2k+1} \sum_{n=1}^{\infty} (-1)^n n^{2k} q^{n^2/2}, \quad \mathcal{V}_2(q) = 2^{2k+1} \sum_{n=1}^{\infty} n^{2k} q^{n^2/2}.$$

Then, for $k \geqslant 1$, *and* $\alpha_{\ell,m,n}, \beta_{\ell,m,n} \in \mathbb{Q}$,

$$\frac{\mathcal{T}_{2k}(q)}{\psi(q)} = \sum_{2\ell+2m+4n=2k} \beta_{\ell,m,n} \mathcal{P}_1^{\ell} e_1^m \mathcal{Q}_1^n,$$

$$\frac{\mathcal{U}_{2k}(q)}{\varphi(-q^{1/2})} = \sum_{2\ell+2m+4n=2k} \alpha_{\ell,m,n} \mathcal{P}_2^{\ell} e_2^m \mathcal{Q}_2^n,$$

$$\frac{\mathcal{V}_{2k}(q)}{\varphi(q^{1/2})} = \sum_{2\ell+2m+4n=2k} \alpha_{\ell,m,n} \mathcal{P}_3^{\ell} e_3^m \mathcal{Q}_3^n.$$

Zeros of Eisenstein series on subgroups of the modular group

Distribution of zeros for Eisenstein series on $SL(2,\mathbb{Z})$

In 1977, Rankin and Swinnerton-Dyer showed

- All of the zeros for the Eisenstein series in a fundamental region for $\mathbb{S}L(2,\mathbb{Z})$ fall on the unit circle between $e^{\pi i/2}$ and $e^{2\pi i/3}$;
- The zeros are equally distributed on the above arc.

In 2008, H. Nozaki proved that the zeros of the Eisenstein series interlace on the aforementioned arc.

The location and distribution of zeros for the Eisenstein series seem to be characteristic of a much wider class of modular forms, including each of the previously discussed Eisenstein series on subgroups of $SL(2,\mathbb{Z})$.

Define

$$Z = e_{1/3}, \quad X(q) = \frac{c^3(q,0)}{e_{1/3}^3(q)}, \quad \text{and} \quad S_{2n}(a) = \sum_{n=1}^{\infty} \frac{n^{2k} q^n}{1+q^n+q^{2n}},$$

where $c(0,q) = 3q\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}$. S. Cooper and Z. -G. Liu prove recursion formulae that provide parametric representations for the cubic Eisenstein series:

$$S_6(q) = \frac{1}{27} z^7 x \left(1 + \frac{4}{3}x\right), \quad S_8(q) = \frac{1}{27} z^9 x \left(1 + 8x + \frac{80}{81}x^2\right),$$

$$S_{10}(q) = \frac{1}{27} z^{11} x \left(1 + 36x + \frac{848}{27}x^2\right),$$

$$S_{12}(q) = \frac{1}{27} z^{13} x \left(1 + \frac{448}{3}x + \frac{12448}{27}x^2 + \frac{6080}{81}x^3\right)$$

$$S_{14}(q) = \frac{1}{27} z^{15} x \left(1 + 604x + \frac{422432}{81}x^2 + \frac{289792}{81}x^3 + \frac{70400}{729}x^4\right).$$

The approximate roots are, respectively,

$$-0.75,$$
$$-7.97301, -0.126991,$$
$$-1.11774, -0.0284857,$$
$$-5.79946, -0.335808, -0.00684074,$$
$$-35.5324, -1.38827, -0.124956, -0.00167997.$$

The roots of these polynomials clearly interlace.

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$$\begin{aligned}
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&-7.97301, -0.126991,\\
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&-35.5324, -1.38827, -0.124956, -0.00167997.
\end{aligned}$$

The roots of these polynomials clearly interlace.

Such interlacing of zeros may be observed in a much wider class of Eisenstein series.

In particular, the corresponding zeros for the Laurent coefficients of the twelve quotients of Jacobian elliptic functions, expressed in terms of the elliptic modulus, are similarly distributed.

The Jacobian Elliptic Functions

Define the functions sn, cn and dn, for $0 < k < 1$, by inverting the relations

$$u = \int_0^{sn(u|k^2)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

$$u = \int_{cn(u|k^2)}^1 \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}} = \int_{dn(u|k^2)}^1 \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}},$$

where $k' = \sqrt{1-k^2}$, $\quad \tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$, and

$$k^2 = \lambda(\tau) = \frac{\theta_3(0 \mid \tau)^4}{\theta_1(0 \mid \tau)^4}, \ \theta_1(0 \mid \tau) = 2\sum_{n=0}^{\infty} q^{\frac{1}{4}(2n+1)^2},$$

$$\theta_3(0,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q = e^{\pi i \tau}.$$

The Jacobian Elliptic Functions

As a function of the parameter τ, the Jacobian elliptic functions each have an analytic continuation to $\mathbb{C} \setminus \{0, 1\}$.

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Define

$$\mathbb{F}_{\Gamma(2)} = \{\tau \in \mathbb{H} \mid |\tau - \frac{1}{2}| > \frac{1}{2}, 0 < \operatorname{Re}\tau < 1\} \\ \cup \{\tau \in \mathbb{H} \mid |\tau + \frac{1}{2}| \geqslant \frac{1}{2}, 0 \leqslant \operatorname{Re}\tau \leqslant 1\}.$$

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For any complex $a \neq 0, 1$, the equation $\lambda(\tau) - a = 0$ has precisely one root in $\mathbb{F}_{\Gamma(2)}$.

For $\tau \in \mathbb{F}_{\Gamma(2)}$, $\lambda(\tau)$ has a single valued inverse given by

$$i \cdot \frac{{}_2F_1(1/2, 1/2, 1; 1 - \lambda)}{{}_2F_1(1/2, 1/2, 1; \lambda)}.$$

The Jacobian Elliptic Functions

Let K denote the complete elliptic integral of the first kind.

Consider the zeros of the series $J_n(q)$ for the 16 series

$$\left(\frac{f}{g}\right)^{\epsilon}(u \mid k^2) = \sum_{n=-\infty}^{\infty} J_n(q)x^n,$$

$$u = \frac{2Kx}{\pi}, \quad \epsilon \in \{1,2\},\ f,g \in \{1, sn, cn, dn\} \text{ and } f \neq g,$$

regarded as functions in the upper half plane, $\tau \in \mathbb{H}$.

In the cases $\epsilon = 2$, $f = 1, cn, dn$, and $g = sn$, the corresponding coefficients, $J_n(q)$, are constant multiples of the classical Eisenstein series,

$$\sum_{m,n=-\infty}^{\infty} (m+n\tau)^{-2k}, \qquad k \geqslant 1.$$

The Jacobian Elliptic Functions

The cases

$$\left(\frac{f}{g}\right)^{\epsilon}(u \mid k^2) = \sum_{n=-\infty}^{\infty} J_n(q)x^n,$$

$$u = \frac{2Kx}{\pi}, \quad \epsilon \in \{1,2\},\ f,g \in \{1, sn, cn, dn\} \text{ and } f \neq g,$$

where we do not have, simultaneously $\epsilon = 2$ and $g = sn$, correspond to modular forms on subgroups intermediate to $SL(2,\mathbb{Z})$ and $\Gamma(2)$:

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, W = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, P = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$\Gamma_0(2) = \langle U, PVP^{-1}\rangle, \quad \Gamma_\theta = \langle V, U^2\rangle, \quad \Gamma^0(2) = \langle W, U^2\rangle.$$

The Jacobian Elliptic Functions

Lemma (Glaisher, 1889)

Let $u = 2Kx/\pi$. *The partial fraction expansion for the Laurent coefficient of each function from* $[\nu^{\epsilon}(u \mid \lambda)]^{(j)}_{x=0}$, *for* j *of appropriate parity, is a nonzero constant multiple of*

$$\sum_{\substack{m,n=-\infty \\ (m,n)\in\mathcal{D}_{\nu}}}^{\infty} \frac{h_{\nu}(m,n)}{(m+n\tau)^{j+\epsilon}},$$

where $\mathcal{D}_{\nu} \subseteq \mathbb{Z}^2$ *and* $h_{\nu}(m,n)$ *are given in the following table.*

The Jacobian Elliptic Functions

ν^{ϵ}	$\mathcal{D}_\nu$	$h_\nu(m,n)$
$K \cdot ns\ u$	$\mathbb{Z}^2$	$(-1)^m$
$K \cdot ds\ u$	$\mathbb{Z}^2$	$(-1)^{m+n}$
$Kk' \cdot nc\ u$	$(\mathbb{Z}/2\mathbb{Z}) \times 2\mathbb{Z}$	$(-1)^{m+n}$
$Kk \cdot sn\ u$	$2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$	$(-1)^m$
$K \cdot dn\ u$	$2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$	$(-1)^n$
$Kk \cdot cd\ u$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(-1)^m$
$(K \cdot ns\ u)^2$ $(K \cdot cs\ u)^2$ $(K \cdot ds\ u)^2$	$\mathbb{Z}^2$	1
$(kK \cdot sn\ u)^2$ $(kK \cdot cn\ u)^2$ $(K \cdot dn\ u)^2$	$2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$	1

The Jacobian Elliptic Functions

v^{ϵ}	$\mathcal{D}_v$	$h_v(m,n)$
$K \cdot cs\ u$	$\mathbb{Z}^2$	$(-1)^n$
$Kk' \cdot sc\ u$	$(\mathbb{Z}/2\mathbb{Z}) \times 2\mathbb{Z}$	$(-1)^n$
$K \cdot dc\ u$	$(\mathbb{Z}/2\mathbb{Z}) \times 2\mathbb{Z}$	$(-1)^m$
$kK \cdot cn\ u$	$2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$	$(-1)^{m+n}$
$kk'K \cdot sd\ u$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(-1)^{m+n}$
$Kk' \cdot nd\ u$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(-1)^n$
$(Kk' \cdot sc\ u)^2$ $(Kk' \cdot nc\ u)^2$ $(K \cdot dc\ u)^2$	$(\mathbb{Z}/2\mathbb{Z}) \times 2\mathbb{Z}$	1
$(kk'K \cdot sd\ u)^2$ $(kK \cdot cd\ u)^2$ $(Kk' \cdot nd\ u)^2$	$(\mathbb{Z}/2\mathbb{Z})^2$	1

Conjecture

Let $u = \frac{2Kx}{\pi}$*. The Laurent coefficient of index* $k+1$ *for each of the given functions* v *about* $x = 0$*, is a modular form on* $\Gamma(2)$ *of weight* k*. The Laurent coefficient of order* k *has* $\lfloor k/2 \rfloor$ *zeros on* $\mathbb{F}_{\Gamma(2)}$*. Each zero is simple and falls in the indicated range.*

v	Location of Zeros
$Kk' \cdot sc\ u, Kk' \cdot nc\ u, K \cdot dc\ u$	$\{(-1+e^{i\theta})/2 \mid 0 \leqslant \theta \leqslant \pi\}$
$Kk \cdot cn\ u, K \cdot dn\ u$	$\{\tau \mid \mathrm{Re}\,\tau = 1\}$
$Kkk' \cdot sd\ u, Kk \cdot cd\ u, Kk' \cdot nd\ u$	$\{\tau \mid \mathrm{Re}\,\tau = 0\}$

Example: $\mathrm{dc}(u \mid \lambda(\tau))$

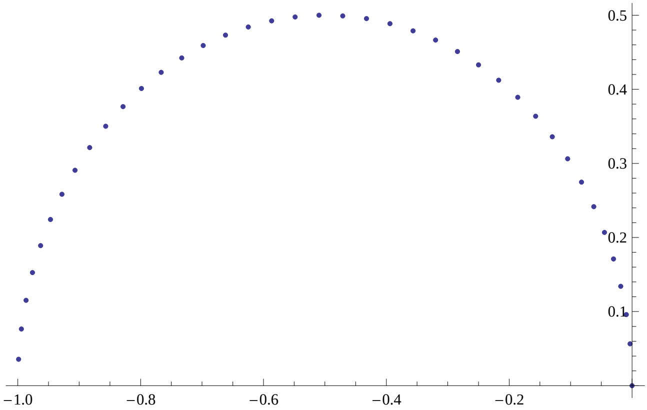

Figure: The zeros for the Maclaurin coefficient of dc of order 80

The Jacobian Elliptic Functions

Conjecture

Let $u = \frac{2Kx}{\pi}$*, and define* $f(k)$ *to be equal to* $\left\lfloor \frac{k}{12} \right\rfloor - 1$ *if* $k \equiv 2 \pmod{12}$ *and* $\left\lfloor \frac{k}{12} \right\rfloor$ *otherwise. The Laurent coefficient of order* $k+1$ *arising from the expansion about* $x = 0$*, for each function*

$$K \cdot \mathrm{ns}(u \mid \lambda(\tau)), \quad K \cdot \mathrm{cs}(u \mid \lambda(\tau)), \quad K \cdot \mathrm{ds}(u \mid \lambda(\tau)), \quad kK \cdot \mathrm{sn}(u \mid \lambda(\tau))$$

has precisely $f(3k)$ τ*-zeros in the respective fundamental domain for its modular group,* Γ_g*.*

The Jacobian Elliptic Functions

Conjecture (continued)

In particular, it is easy to show that

$$\Gamma_{ds} = \Gamma_{\theta}, \quad \Gamma_{ns} = \Gamma_0(2), \quad \textit{and} \quad \Gamma_{sn} = \Gamma_{cs} = \Gamma^0(2).$$

The τ*-zeros of the Laurent coefficients of* $kK \cdot sn\ u$ *in a fundamental region are all simple and located on*

$$\{\tau \in \mathbb{H} \mid \operatorname{Re}\tau = 1, \operatorname{Im}\tau > 1/2\}.$$

The τ*-zeros of the Laurent coefficients for* ds, cs, ns *in a fundamental region for* Γ_{θ} *are all simple and located on*

$$\{\tau \in \mathbb{H} \mid |\tau| = 1, -1 \leqslant \operatorname{Re}\tau \leqslant 0\}.$$

The $SL(2,\mathbb{Z})$ Equivalence of the Zeros

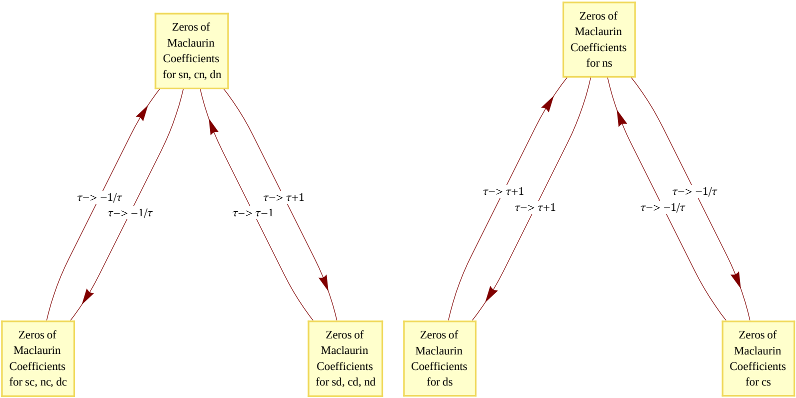

Figure: The $SL(2,\mathbb{Z})$-equivalence of the zeros of Maclaurin coefficients for the Jacobian elliptic functions

Conjectured Interlacing Properties

Let

- $u = \frac{2Kx}{\pi}$;
- $[v]_{x=0}^{(n)}$ denote the Maclaurin coefficient of index n for the function v about $x = 0$;
- $\{a_j \mid j = 1, 2 \ldots, n\}$ denote the imaginary parts of the τ-zeros in $\mathbb{F}_{\Gamma(2)}$ a given $[v]_{x=0}^{(2n-1)}$, $[w]_{x=0}^{(2n)}$, where

$$v \in \{Kk \cdot sn(u \mid \lambda(\tau))\},$$
$$w \in \{Kk \cdot cn(u \mid \lambda(\tau)), K \cdot dn(u \mid \lambda(\tau)), Kk'k \cdot sd(u \mid \lambda(\tau)),$$
$$Kk \cdot cd(u \mid \lambda(\tau)), Kk' \cdot nd(u \mid \lambda(\tau))$$

- $\{b_j \mid j = 1, 2 \ldots, n\}$ denote the respective imaginary parts of τ-zeros for the corresponding function $[v]_{x=0}^{(2n+1)}$, $[v]_{x=0}^{(2n+2)}$.

Then $b_j < a_j < b_{j+1}$.

Remarks

- I suggest that the remaining Jacobian elliptic functions (and their squares) have Laurent coefficients whose zeros are similarly distributed on appropriate arcs in $\mathbb{H}$.
- Rankin and Swinnterton-Dyer's argument can be used to show that approximately $1/3$ of the zeros of each series under consideration fall on the appropriate arcs.
- Provided these zeros are simple, and that there are no other zeros, Nozaki's argument can be applied to show that the zeros interlace on these arcs.
- S. Garthwaite, L. Long, H. Swisher, S. Treneer have interesting preliminary results that extend the work Rankin and Swinnterton-Dyer to certain modular forms on congruence subgroups. They are able to locate $\geqslant 90\%$ of the zeros.

Divisor Polynomials

To study the zeros numerically, we use the well known Maclaurin expansions

$$\begin{aligned}
\mathrm{sn}(u \mid \lambda) &= u - (1+\lambda)\frac{u^3}{3!} + (1 + 14\lambda + \lambda^2)\frac{u^5}{5!} + \cdots \\
\mathrm{cn}(u \mid \lambda) &= 1 - \frac{u^2}{2!} + (1+4\lambda)\frac{u^4}{4!} + \cdots \\
\mathrm{dn}(u \mid \lambda) &= 1 - \frac{\lambda u^2}{2!} + (4\lambda + \lambda^2)\frac{u^4}{4!} + \cdots ,
\end{aligned}$$

then approximate the zeros of the corresponding polynomials in λ and map the zeros to $\mathbb{F}_{\Gamma(2)}$ via

$$i \cdot \frac{{}_2F_1(1/2, 1/2, 1; 1-\lambda)}{{}_2F_1(1/2, 1/2, 1; \lambda)}.$$

Conjectured properties for divisor polynomials

Let $P_{f,n}(\lambda)$ denote the nth Laurent coefficient of the Jacobian elliptic function f expressed as a polynomial over $\mathbb{Z}$ in the elliptic modulus λ.

- For $n \geqslant 1$, all zeros of

$$P_{sd,2n-1}(\lambda), \quad P_{cd,2n}(\lambda), \quad \text{and} \quad P_{nd,2n}(\lambda)$$

  lie in $[0, 1]$, and the zeros of successive polynomials interlace.
- For $n \geqslant 1$, all zeros of

$$P_{sc,2n-1}(\lambda), \quad P_{dc,2n}(\lambda), \quad \text{and} \quad P_{nc,2n}(\lambda)$$

  lie in $[1, \infty)$, and the zeros of successive polynomials interlace.

Conjectured properties for divisor polynomials

- For $n \geqslant 1$, all zeros of

$$P_{sn,2n-1}(\lambda), \quad P_{cn,2n}(\lambda), \quad \text{and} \quad P_{dn,2n}(\lambda)$$

lie in $(-\infty, 0]$, and the zeros of successive polynomials interlace.

- For $n \geqslant 1$, the zeros of

$$P_{ns,2n-1}(\lambda), \quad P_{cs,2n}(\lambda), \quad \text{and} \quad P_{ds,2n}(\lambda)$$

are contained in the union of the following sets:

$$\{-e^{i\theta} \mid -2\pi/3 \leqslant \theta \leqslant 2\pi/3\}, \{e^{i\theta} + 1 \mid -2\pi/3 \leqslant \theta \leqslant 2\pi/3\},$$
$$\{\operatorname{Re} z = \frac{1}{2}, -\sqrt{3}/2 \leqslant \operatorname{Im} z \leqslant \sqrt{3}/2\}.$$

On each of these arcs, the zeros of successive polynomials interlace.

Corollary

For $n \geqslant 0$, the $(2n+1)$th Maclaurin coefficient of $\mathrm{sd}(u \mid x)$ is a polynomial of degree n in which each coefficient is a nonzero integer. The sign of the k^{th} coefficient in the polynomial is $(-1)^{n+k}$.

This follows from the differential equation

$$\frac{d^2}{du^2}\,\mathrm{sd}(u \mid x) = 2x(x-1)\,\mathrm{sd}^3(u \mid x) + (2x-1)\,\mathrm{sd}(u \mid x)$$

and the corresponding recursion formula for the Maclaurin coefficients.

Corollary

Let $P_{n,sd}(x)$ denote the Maclaurin coefficient of $\mathrm{sd}(u \mid x)$ of order n expressed in terms of the square of the elliptic modulus x. Then

$$P_{4n+1,sd}(x) = P_{4n+1,sd}(1-x) \qquad \textit{and} \qquad P_{4n+3,sd}(x) = -P_{4n+3,sd}(1-x).$$

In particular, if α is a root of $P_{2n-1,sd}(x)$, then $1-\alpha$ is also a root.

This follows from Jacobi's rotation formula.

Corollary

Any real zeros of $P_{2n-1,sd}(x)$ are located in $[0,1]$. Moreover, the set of real zeros of $P_{2n-1,sd}(x)$ is nonempty.

This follows from the preceeding Corollaries and Rankin, Swinnterton-Dyer's argument.

Divisor Polynomials corresponding to $\mathrm{sd}(u \mid \lambda)$

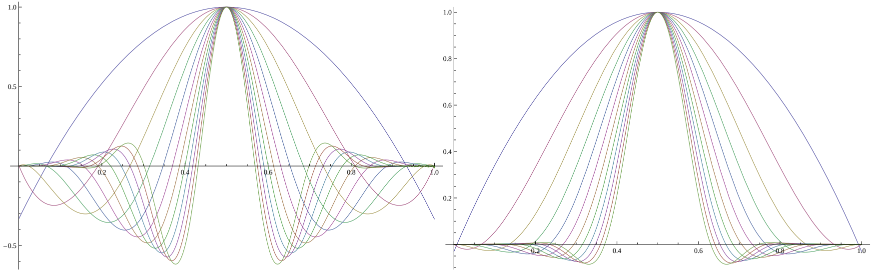

Figure: The normalized divisor polynomials on $[0, 1]$ for the Maclaurin coefficients of $\mathrm{sd}(u \mid \lambda)$.

Divisor Polynomials corresponding to $\mathrm{sd}(u \mid \lambda)$

Theorem

If $n \equiv 1 \pmod 4$,

$$\left[\frac{d}{dx} P_{n,sd}(x)\right]_{x=1/2} = 0.$$

If $n \equiv 3 \pmod 4$,

$$\lim_{x \to 1/2} \frac{d}{dx} \left(\frac{P_{n,sd}(x)}{x - 1/2} \right) = 0.$$

Divisor Polynomials corresponding to $\mathrm{sd}(u \mid \lambda)$

Theorem

If $n \equiv 1 \pmod 4$,

$$\max_{x \in [0,1]} |P_{n,sd}(x)| = |P_{n,sd}(1/2)|.$$

If $n \equiv 3 \pmod 4$,

$$\max_{x \in [0,1]} |(x - 1/2)^{-1} P_{n,sd}(x)| = \lim_{x \to 1/2} |(x - 1/2)^{-1} P_{n,sd}(x)|.$$

Connection to orthogonal polynomials

We may apply the integral formula

$$u = \int_0^{\mathrm{sd}(u|x)} (1-(1-x)t^2)^{-1/2}(1+xt^2)^{-1/2}dt \qquad (5)$$

and use the generating function the for the Legendre polynomials P_n

$$\frac{1}{\sqrt{1-2rt+t^2}} = \sum_{n=0}^{\infty} P_n(r)t^n$$

to write the integrand in (5) in the form

$$g(t) := \sum_{n=0}^{\infty} P_n\left(\frac{1-2x}{2x^{1/2}(x-1)^{1/2}}\right) x^{n/2}(x-1)^{n/2}t^{2n}.$$

Connection to orthogonal polynomials

Define

$$g_{2n} := \left[\frac{d^{2n}}{dt^{2n}} g(t)\right]_{t=0} = (2n)! P_n\left(\frac{1-2x}{2x^{1/2}(x-1)^{1/2}}\right) x^{n/2}(x-1)^{n/2},$$

and take iterated derivatives of the identity on the previous slide, to conclude that

$$\frac{1}{3!}\left[\frac{d^3}{du^3}\,\mathrm{sd}(u \mid x)\right]_{u=0} = -g_2,$$

$$\frac{1}{5!}\left[\frac{d^5}{du^5}\,\mathrm{sd}(u \mid x)\right]_{u=0} = 10g_2 - g_4,$$

$$\frac{1}{7!}\left[\frac{d^7}{du^7}\,\mathrm{sd}(u \mid x)\right]_{u=0} = -280g_2^3 + 56g_2g_4 - g_6,$$

$$\frac{1}{9!}\left[\frac{d^9}{du^9}\,\mathrm{sd}(u \mid x)\right]_{u=0} = 15400g_2^4 - 4620g_2^2g_4 + 126g_4^2 + 120g_2g_6 - g_8.$$

Remarks

- The polynomials corresponding to the remaining Jacobian elliptic functions and their squares satisfy similar properties.
- Brillhart and Lomont have shown that $1/2$ is a zero of $P_{4n+3,sd}(x)$ and that $(2 \pm \sqrt{3})/4$ are each zeros of $P_{6n+5,sd}(x)$ for each $n \geqslant 0$.
- Numerical calculations suggest that, apart from the factors corresponding to the zeros of Brillhart and Lomont, the polynomials are irreducible and their Galois group is the full symmetric group.