Venkatachaliengar's proof of the transformation formula for the eta-function, and some extensions.

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Outline

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2. Ramanujan's function k

3. Elliptic functions and $\Gamma_0(10)$

4. Other extensions and $\Gamma_0(p)$

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Development of Elliptic Functions
according to Ramanujan

K. Venkatachaliengar, 1988(?)

1. The basic identity

2. The differential equations of P, Q and R

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plus three appendices. viii + 147 pp.

The eta-function

- Definition

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{j=1}^{\infty} (1 - e^{2\pi i j \tau}) = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j),$$

Im$(\tau) > 0$ and $q = \exp(2\pi i \tau)$, so $|q| < 1$.

- Transformation formula

$$\eta(\tau) = \sqrt{\frac{i}{\tau}}\, \eta\left(-\frac{1}{\tau}\right).$$

- Equivalent transformation formula

$$q^{1/24} \prod_{j=1}^{\infty} (1 - q^j) = \sqrt{\frac{1}{t}}\, q_1^{1/24} \prod_{j=1}^{\infty} (1 - q_1^j),$$

$$q = e^{-2\pi t}, \qquad q_1 = e^{-2\pi/t}$$

$$\tau = it, \qquad \mathrm{Re}(t) > 0.$$

Venkatachaliengar's proof [pp. 33–35]

Supose $\mathrm{Im}(\tau) > 0$, $q = e^{2\pi i\tau}$ and define

$$\phi(z|\tau) = \frac{1}{4}\cot\frac{z}{2} + \sum_{j=1}^{\infty} \frac{q^j}{1-q^j} \sin jz.$$

- $\phi(z|\tau)$ is an odd function of z

- $\phi(z|\tau)$ analytic except for simple poles at $z = 2\pi m + 2\pi n\tau$, $m, n \in \mathbb{Z}$.
  The residue at each pole is 1/2.

- $\phi(z + 2\pi|\tau) = \phi(z|\tau)$

  $\phi(z + 2\pi\tau|\tau) = \phi(z|\tau) - \frac{i}{2}$.

Transformation formula

- The functions $\phi(z|\tau)$ and $\frac{1}{\tau}\phi\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right)$ have the same poles and residues.

- The difference $f(z) = \phi(z|\tau) - \frac{1}{\tau}\phi\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right)$

  is entire and has the properties

$$f(z+2\pi) = f(z) + \frac{1}{2i\tau},$$

$$f(z+2\pi\tau) = f(z) + \frac{1}{2i}.$$

- The function $\phi(z|\tau) - \frac{1}{\tau}\phi\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) - \frac{1}{4\pi i\tau}z$

  is odd, entire and doubly periodic.
  By Liouville's theorem it is identically zero.

Series expansion

$$\begin{aligned}\phi(z|\tau) &= \frac{1}{4}\cot\frac{z}{2} + \sum_{j=1}^{\infty}\frac{q^j}{1-q^j}\sin jz \\ &= \frac{1}{2z} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}S_{2n-1}(\tau)z^{2n-1}\end{aligned}$$

where

$$S_{2n-1}(\tau) = -\frac{B_{2n}}{4n} + \sum_{j=1}^{\infty}\frac{j^{2n-1}q^j}{1-q^j}$$

and B_{2n} are the Bernoulli numbers defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}t^n.$$

In particular,

$$S_1(\tau) = -\frac{1}{24} + \sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}.$$

A transformation formula

Substitute the series expansion for $\phi(z|\tau)$ into the transformation formula and equate coefficients of z:

$$\begin{aligned} S_1(\tau) &= \frac{1}{\tau^2} S_1\left(-\frac{1}{\tau}\right) - \frac{1}{4\pi i\tau}, \\ S_{2n-1}(\tau) &= \frac{1}{\tau^{2n}} S_{2n-1}\left(-\frac{1}{\tau}\right), \quad n \geq 2. \end{aligned}$$

The result for S_1 may be written in the form

$$\begin{aligned} & 1 - 24 \sum_{j=1}^{\infty} \frac{je^{-2\pi jt}}{1-e^{-2\pi jt}} \\ &= -\frac{1}{t^2}\left(1 - 24 \sum_{j=1}^{\infty} \frac{je^{-2\pi j/t}}{1-e^{-2\pi j/t}}\right) - \frac{6}{\pi t}. \end{aligned}$$

The value $t = 1$ solves a problem posed by Ramanujan as Q. 387 in the Journal of the Indian Mathematical Society:

$$\sum_{j=1}^{\infty} \frac{j}{e^{2\pi j} - 1} = \frac{1}{24} - \frac{1}{8\pi}.$$

Venkatachaliengar's proof: conclusion

$$1 - 24\sum_{j=1}^{\infty}\frac{je^{-2\pi jt}}{1-e^{-2\pi jt}}$$
$$= \ -\frac{1}{t^2}\left(1-24\sum_{j=1}^{\infty}\frac{je^{-2\pi j/t}}{1-e^{-2\pi j/t}}\right)-\frac{6}{\pi t}.$$

- Integrate, then exponentiate:

$$q^{1/12}\prod_{j=1}^{\infty}(1-q^j)^2=\frac{A}{t}\,q_1^{1/12}\prod_{j=1}^{\infty}(1-q_1^j)^2,$$

$$q=e^{-2\pi t},\qquad q_1=e^{-2\pi/t},\qquad t>0$$

and A is a constant, independent of t.

- Set $t=1$ to get $A=1$.

- Proved for $t>0$. It holds for $\mathrm{Re}(t)>0$ by analytic continuation.

Exercise

Integrate with respect to z:

$$\phi(z|\tau) = \frac{1}{\tau}\phi\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) + \frac{1}{4\pi i\tau}z$$

to deduce the transformation formula for the Jacobian theta function $\theta_1(z|\tau)$.

Part 2: Ramanujan's function k

- $R(q) = q^{1/5} \prod_{j=1}^{\infty} \frac{(1-q^{5j-4})(1-q^{5j-1})}{(1-q^{5j-3})(1-q^{5j-2})}$
  Rogers-Ramanujan continued fraction.

- $k = R(q)R^2(q^2) = q \prod_{j=1}^{\infty} (1-q^j)^{-c(j)}$

$$
\begin{aligned}
c(j) &= (-1)^j \left(\frac{j}{5}\right) \\
&= \begin{cases} 1 & \text{if } j \equiv 3,4,6,7 \pmod{10}, \\ -1 & \text{if } j \equiv 1,2,8,9 \pmod{10}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

- Properties of k given by Ramanujan in the lost notebook have been analyzed by S. S. Rangachari and S. Raghavan, S.-Y. Kang, G. E. Andrews and B. C. Berndt, and C. Gugg.

Extending Ramanujan's results for k

- $k = q \prod_{j=1}^{\infty} (1 - q^j)^{-c(j)}$

- $z = q \frac{d}{dq} \log k = 1 + \sum_{j=1}^{\infty} (-1)^j \left(\frac{j}{5}\right) \frac{jq^j}{1 - q^j}$

- Theorem: Each of z, $\frac{1}{k} - k$, $\frac{1}{k} + 1 - k$ and $\frac{1}{k} - 4 - k$ have simple expressions in terms of eta-quotients.

- Equivalently, each of $\eta^{24}(\tau)$, $\eta^{24}(2\tau)$, $\eta^{24}(5\tau)$ and $\eta^{24}(10\tau)$ is expressible as

$$z^6 \times \text{rational function of } k.$$

k and eta-quotients

Let

$$\eta_n = \eta(n\tau) = q^{n/24} \prod_{j=1}^{\infty} (1 - q^{nj}).$$

$$\begin{aligned}
\eta_1^{24} &= z^6 \frac{k(1-4k-k^2)^4}{(1-k^2)^4(1+k-k^2)}, \\
\eta_2^{24} &= z^6 \frac{k^2(1+k-k^2)^4}{(1-k^2)^5(1-4k-k^2)}, \\
\eta_5^{24} &= z^6 \frac{k^5(1-k^2)^4}{(1+k-k^2)^5(1-4k-k^2)^4}, \\
\eta_{10}^{24} &= z^6 \frac{k^{10}}{(1-k^2)(1+k-k^2)^4(1-4k-k^2)^5}.
\end{aligned}$$

Ramanujan's Eisenstein series

- Let

$$P(q) = 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j},$$

$$Q(q) = 1 + 240 \sum_{j=1}^{\infty} \frac{j^3 q^j}{1-q^j},$$

$$R(q) = 1 - 504 \sum_{j=1}^{\infty} \frac{j^5 q^j}{1-q^j}.$$

  These are the first three coefficients in the expansion of $\phi(z|\tau)$ in Venkatachaliengar's proof.

- Note: $q\frac{d}{dq}\log \eta^{24}(\tau) = P(q)$.

Ramanujan's Eisenstein series $P(q)$

- Theorem:

$$\begin{pmatrix} P_1 \\ P_2 \\ P_5 \\ P_{10} \end{pmatrix} = \begin{pmatrix} 4 & 1 & -4 & 6 \\ \frac{5}{2} & -2 & \frac{1}{2} & 3 \\ -\frac{4}{5} & 1 & \frac{4}{5} & \frac{6}{5} \\ \frac{1}{10} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{(1+k^2)}{(1-k^2)} z \\ \frac{(1+k^2)}{(1+k-k^2)} z \\ \frac{(1+k^2)}{(1-4k-k^2)} z \\ k\frac{dz}{dk} \end{pmatrix}.$$

where $P_n = P(q^n)$.

- Proof: Apply logarithmic differentiation to the corresponding results for η_1, η_2, η_5 and η_{10}.

Ramanujan's Eisenstein series

- Each of $P(q)$, $P(q^2)$, $P(q^5)$ and $P(q^{10})$ may be expressed in the form

$$z \times \text{rational function of } k + \text{const} \times k\frac{dz}{dk}.$$

- Each of $Q(q)$, $Q(q^2)$, $Q(q^5)$ and $Q(q^{10})$ may be expressed in the form

$$z^2 \times \text{rational function of } k.$$

- Each of $R(q)$, $R(q^2)$, $R(q^5)$ and $R(q^{10})$ may be expressed in the form

$$z^3 \times \text{rational function of } k.$$

- Analogue of a catalogue for classical theta functions given by Ramanujan in Chapter 17 of his second notebook.

Chapter 19 of Ramanujan's 2nd notebook

$$1+\sum_{j=1}^{\infty}(-1)^j\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j}$$
$$=\ \frac{1}{4}\varphi(-q)\varphi(-q^5)\left(5\varphi^2(-q^5)-\varphi^2(-q)\right),$$

$$\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^{2j}}$$
$$=\ q\psi(q)\psi(q^5)\left(\psi^2(q)-5q\psi^2(q^5)\right),$$

where

$$\varphi(q)=\sum_{j=-\infty}^{\infty}q^{j^2}\quad\text{and}\quad\psi(q)=\sum_{j=0}^{\infty}q^{j(j+1)/2}.$$

Each factor on the right hand sides of these identities may be written as an eta-quotient.

Connection with k and z

- $$1+\sum_{j=1}^{\infty}(-1)^j\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j}=z,$$

- $$\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^{2j}}=\frac{zk}{1-k^2},$$

- $$\sum_{j=1}^{\infty}(-1)^{j-1}\frac{jq^j(1-q^j)(1-q^{2j})}{1-q^{5j}}=\frac{zk}{1+k-k^2},$$

- $$\sum_{j=1}^{\infty}\frac{jq^j(1-q^{2j})(1-q^{6j})}{1-q^{10j}}=\frac{zk}{1-4k-k^2}.$$

Part 3: Elliptic functions

This section is based on joint work with Heung Yeung Lam (preprint).

Generalized Eisenstein series

For any positive integer n, define

$$\begin{aligned}
F_1(2n|\tau) &= \frac{B_{2n,10}}{4n} + \sum_{j=1}^{\infty} (-1)^j \left(\frac{j}{5}\right) \frac{j^{2n-1} q^j}{1-q^j}, \\
F_2(2n|\tau) &= \sum_{j=1}^{\infty} \left(\frac{j}{5}\right) \frac{j^{2n-1} q^j}{1-q^{2j}}, \\
F_5(2n|\tau) &= \sum_{j=1}^{\infty} (-1)^{j-1} \frac{j^{2n-1} q^j (1-q^j)(1-q^{2j})}{1-q^{5j}}, \\
F_{10}(2n|\tau) &= \sum_{j=1}^{\infty} \frac{j^{2n-1} q^j (1-q^{2j})(1-q^{6j})}{1-q^{10j}}.
\end{aligned}$$

- $B_{2n,10}$ are generalized Bernoulli numbers, defined by

$$x\left(\frac{e^{4x}+e^{3x}-e^{2x}-e^{x}}{e^{5x}+1}\right) = \sum_{n=0}^{\infty} B_{n,10} \frac{x^n}{n!}.$$

- $B_{0,10} = 0$, $B_{2,10} = 4$, $B_{4,10} = -8 \times 17$, $B_{6,10} = 12 \times 871$, and $B_{8,10} = -16 \times 92777$.

Elliptic functions

$$F_2(2n|\tau) = \sum_{j=1}^{\infty} \left(\frac{j}{5}\right) \frac{j^{2n-1} q^j}{1 - q^{2j}}.$$

$$M_2(z|\tau)$$

$$= \sum_{n=1}^{\infty} F_2(2n|\tau) \frac{(-1)^{n-1}(2z)^{2n-1}}{(2n-1)!}$$

$$= \sum_{j=1}^{\infty} \left(\frac{j}{5}\right) \frac{q^j}{1 - q^{2j}} \sin 2jz$$

$$= \frac{1}{2i} \sum_{j=-\infty}^{\infty} \frac{\left(\begin{array}{l} q^{2j-1} e^{2iz} - q^{4j-2} e^{4iz} \\ -q^{6j-3} e^{6iz} + q^{8j-4} e^{8iz} \end{array} \right)}{1 - q^{10j-5} e^{10iz}}$$

$$= \frac{\eta(2\tau)\eta(5\tau)}{2\eta(\tau)} \times \frac{\theta(z|\tau)\theta(2z|2\tau)\theta(5z|10\tau)}{\theta(z|2\tau)\theta(5z|5\tau)}.$$

Periods and irreducible sets of zeros and poles

Function	Periods (ω_1, ω_2)	Poles
$M_1(z\|\tau)$	$(\pi, \pi\tau)$	$\frac{j\omega_1}{10}$, $j \in \{1,3,7,9\}$
$M_2(z\|\tau)$	$(\pi, 2\pi\tau)$	$\frac{j\omega_1}{5} + \frac{\omega_2}{2}$, $j \in \{1,2,3,4\}$
$M_5(z\|\tau)$	$(\pi, 5\pi\tau)$	$\frac{\omega_1}{2} + \frac{j\omega_2}{5}$, $j \in \{1,2,3,4\}$
$M_{10}(z\|\tau)$	$(\pi, 10\pi\tau)$	$\frac{j\omega_2}{10}$, $j \in \{1,3,7,9\}$

Each function has zeros at 0, $\omega_1/2$, $\omega_2/2$ and $(\omega_1 + \omega_2)/2$.

The order of each elliptic function is 4.

Pole sets

Let $u,\ v \in \mathbb{C}$ with $\mathrm{Im}(v/u) > 0$. Let

$$
\begin{aligned}
P_1(u,v) &= \left\{\left(m+\frac{r}{10}\right)u+nv\right\} \\
P_2(u,v) &= \left\{\left(m+\frac{s}{5}\right)u+\left(n+\frac{1}{2}\right)v\right\} \\
P_5(u,v) &= \left\{\left(m+\frac{1}{2}\right)u+\left(n+\frac{s}{5}\right)v\right\} \\
P_{10}(u,v) &= \left\{mu+\left(n+\frac{r}{10}\right)v\right\}
\end{aligned}
$$

where

$m,\ n \in \mathbb{Z}$, $r \in \{1,3,7,9\}$ and $s \in \{1,2,3,4\}$.

Thus $P_k(u,v)$ is the sets of poles of the function $M_k\left(\frac{\pi z}{u}\middle|\frac{v}{ku}\right)$, $k \in \{1,2,5,10\}$.

Intra-relations

- Let $u,\ v \in \mathbb{C}$ with $\mathrm{Im}(v/u) > 0$.

- Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$

- Let $V = av + bu$, $U = cv + du$.

- $c \equiv 0 \pmod{10} \Rightarrow P_1(u, v) = P_1(U, V)$.

- $\left\{ \begin{array}{c} b \equiv 0 \pmod{2} \\ c \equiv 0 \pmod{5} \end{array} \right\} \Rightarrow P_2(u, v) = P_2(U, V)$.

- $\left\{ \begin{array}{c} b \equiv 0 \pmod{5} \\ c \equiv 0 \pmod{2} \end{array} \right\} \Rightarrow P_5(u, v) = P_5(U, V)$.

- $b \equiv 0 \pmod{10} \Rightarrow P_{10}(u, v) = P_{10}(U, V)$.

Intra-relations: example

- Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$,
  $b \equiv 0 \pmod 2$ and $c \equiv 0 \pmod 5$.

- Let u, $v \in \mathbb{C}$ with $\mathrm{Im}(v/u) > 0$.
  Let $V = av + bu$, $U = cv + du$.

- $\dfrac{1}{U} M_2\left(\dfrac{\pi z}{U}\middle|\dfrac{V}{2U}\right) = \left(\dfrac{d}{5}\right) \dfrac{1}{u} M_2\left(\dfrac{\pi z}{u}\middle|\dfrac{v}{2u}\right).$

- Proof: $M_2\left(\dfrac{\pi z}{U}\middle|\dfrac{V}{2U}\right)$ and $M_2\left(\dfrac{\pi z}{u}\middle|\dfrac{v}{2u}\right)$ are elliptic functions with the same periods, zeros and poles. Their quotient is therefore a constant, which can be determined by examining the behavior at the pole

$$z = \frac{U}{5} + \frac{V}{2} = \frac{du}{5} + \frac{av}{2}.$$

Intra-relations: conclusion

- Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}(2,\mathbb{Z})$, $c \equiv 0 \pmod{10}$.

- For $k \in \{1,2,5,10\}$, the elliptic functions satisfy

$$M_k\left(z\left|\frac{a\tau+b}{c\tau+d}\right.\right) = \left(\frac{d}{5}\right)(c\tau+d)M_k\left((c\tau+d)z|\tau\right)$$

and the Eisenstein series satisfy

$$F_k\left(2n\left|\frac{a\tau+b}{c\tau+d}\right.\right) = \left(\frac{d}{5}\right)(c\tau+d)^{2n}F_k\left(2n|\tau\right).$$

- Example:

$$\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^{2j}} = \left(\frac{d}{5}\right)(c\tau+d)^2\sum_{j=1}^{\infty}\frac{jq_1^j}{1-q_1^j},$$

$$q = \exp(2\pi i\tau),\ q_1 = \exp\left(2\pi i\left(\frac{a\tau+b}{c\tau+d}\right)\right).$$

Inter-relations

- Let $u_1,\ v_1 \in \mathbb{C}$ with $\text{Im}(v_1/u_1) > 0$.

- $\begin{pmatrix} v_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ u_1 \end{pmatrix},$

- $\begin{pmatrix} v_5 \\ u_5 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ u_1 \end{pmatrix},$

- $\begin{pmatrix} v_{10} \\ u_{10} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ u_1 \end{pmatrix}.$

- $P_1(u_1 v_1) = P_2(u_2, v_2)$

$$= P_5(u_5, v_5) = P_{10}(u_{10}, v_{10}).$$

Inter-relations among elliptic functions

$$
\begin{aligned}
&M_1(z|\tau)\\
&= \frac{-1}{(5\tau-2)} M_2\left(\frac{z}{5\tau-2}\middle|\frac{2\tau-1}{2(5\tau-2)}\right)\\
&= \frac{1}{\sqrt{5}(4\tau+1)} M_5\left(\frac{z}{4\tau+1}\middle|\frac{5\tau+1}{5(4\tau+1)}\right)\\
&= \frac{-1}{\sqrt{5}\,\tau} M_{10}\left(\frac{z}{\tau}\middle|\frac{-1}{10\tau}\right).
\end{aligned}
$$

Atkin-Lehner involutions

$$
\begin{aligned}
&F_1(2n|\tau)\\
&= \frac{-1}{(5\tau-2)^{2n}} F_2\left(2n\middle|\frac{2\tau-1}{2(5\tau-2)}\right)\\
&= \frac{1}{\sqrt{5}(4\tau+1)^{2n}} F_5\left(2n\middle|\frac{5\tau+1}{5(4\tau+1)}\right)\\
&= \frac{-1}{\sqrt{5}\,\tau^{2n}} F_{10}\left(2n\middle|\frac{-1}{10\tau}\right).
\end{aligned}
$$

Inter-relations: example

$$1+\sum_{j=1}^{\infty}(-1)^j\left(\frac{j}{5}\right)\frac{jq_1^j}{1-q_1^j}$$

$$=\ \frac{-1}{(5\tau-2)^2}\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq_2^j}{1-q_2^{2j}}$$

$$\begin{aligned} q_1 &= \exp(2\pi i\tau), \\ q_2 &= \exp\left(2\pi i\frac{(2\tau-1)}{2(5\tau-2)}\right). \end{aligned}$$

Atkin-Lehner relations

$$q_1 = \exp(2\pi i\tau), \quad q_2 = \exp\left(2\pi i \frac{(2\tau-1)}{2(5\tau-2)}\right).$$

$$k_1 = k(q_1), \qquad k_2 = k(q_2),$$

$$\begin{pmatrix} \frac{k_1}{1+k_1^2} \\ \frac{1-k_1^2}{1+k_1^2} \\ \frac{1+k_1-k_1^2}{1+k_1^2} \\ \frac{1-4k_1-k_1^2}{1+k_1^2} \end{pmatrix} = A_r \begin{pmatrix} \frac{k_2}{1+k_2^2} \\ \frac{1-k_2^2}{1+k_2^2} \\ \frac{1+k_2-k_2^2}{1+k_2^2} \\ \frac{1-4k_2-k_2^2}{1+k_2^2} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

Part 4. Other extensions: $\Gamma_0(p)$

- Let p be an odd prime.

- $\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \middle| c \equiv 0 (\mathrm{mod}\ p) \right\}$

- Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$.

- Let $w_1,\ w_2 \in \mathbb{C}$ with $\mathrm{Im}(w_2/w_1) > 0$.

- Let $w_4 = aw_2 + bw_1$ and $w_3 = cw_2 + dw_1$.

- Let $\tau = \dfrac{w_2}{w_1}$.

- Then $\dfrac{w_4}{w_3} = \dfrac{a\tau + b}{c\tau + d}$ and $\dfrac{w_3}{w_1} = c\tau + d$.

Lattice subsets

- $\Lambda^{\pm}(w_1, w_2) = \left\{ m w_1 + n w_2 : \left(\frac{n}{p}\right) = \pm 1 \right\}$,

- $\Omega^{\pm}(w_1, w_2) = \left\{ \frac{m}{p} w_1 + w_2 : \left(\frac{m}{p}\right) = \pm 1 \right\}$.

- Intra-relationships:

$$\Lambda^{\pm}(w_1, w_2) = \Lambda^{\pm\left(\frac{d}{p}\right)}(w_3, w_4)$$

$$\Omega^{\pm}(w_1, w_2) = \Omega^{\pm\left(\frac{d}{p}\right)}(w_3, w_4)$$

- Inter-relationship:

$$\Lambda^{\pm}(w_1, w_2) = \Omega^{\pm\left(\frac{d}{p}\right)}(p w_2, -w_1)$$

Modular relations for $\Gamma_0(p)$

Idea:

- Cook up an elliptic function with zero set given by $\Lambda^+(w_1, w_2)$ and pole set given by $\Lambda^-(w_1, w_2)$, and another one with zero set $\Omega^+(w_1, w_2)$ and pole set $\Omega^-(w_1, w_2)$.

- Exploit the intra-relationships.

- Exploit the inter-relationships.

Construction of elliptic functions

$$\begin{aligned} R &= \left\{ r : 1 \le r \le p-1,\ \left(\frac{r}{p}\right) = 1 \right\} \\ NR &= \left\{ r : 1 \le r \le p-1,\ \left(\frac{r}{p}\right) = -1 \right\} \end{aligned}$$

$$\theta(z|\tau) = 2q^{1/8}\sin z \times \prod_{n=1}^{\infty}(1-q^n e^{2iz})(1-q^n e^{-2iz})(1-q^n).$$

$$F(z;\omega_1,\omega_2) = \frac{\prod_{r\in\mathrm{R}} \exp\left(\frac{-2\pi i r z}{p\omega_1}\right)\theta_1\left(\frac{\pi}{\omega_1}(z-r\omega_2)\ \middle|\ \frac{p\omega_2}{\omega_1}\right)}{\prod_{r\in\mathrm{NR}} \exp\left(\frac{-2\pi i r z}{p\omega_1}\right)\theta_1\left(\frac{\pi}{\omega_1}(z-r\omega_2)\ \middle|\ \frac{p\omega_2}{\omega_1}\right)},$$

$$G(z;\omega_1,\omega_2) = \frac{\prod_{r\in\mathrm{R}} \theta_1\left(\pi\left(\frac{z}{\omega_1}-\frac{r}{p}\right)\ \middle|\ \frac{\omega_2}{\omega_1}\right)}{\prod_{r\in\mathrm{NR}} \theta_1\left(\pi\left(\frac{z}{\omega_1}-\frac{r}{p}\right)\ \middle|\ \frac{\omega_2}{\omega_1}\right)}.$$

Transformation formulas

- Intra-relationships:

$$\frac{F(z;\omega_1,\omega_2)}{F(0;\omega_1,\omega_2)} = \left(\frac{F(z;\omega_3,\omega_4)}{F(0;\omega_3,\omega_4)}\right)^{\left(\frac{d}{p}\right)},$$

$$\frac{G(z;\omega_1,\omega_2)}{G(0;\omega_1,\omega_2)} = \left(\frac{G(z;\omega_3,\omega_4)}{G(0;\omega_3,\omega_4)}\right)^{\left(\frac{d}{p}\right)},$$

- Inter-relationship:

$$\frac{F(z;\omega_1,\omega_2)}{F(0;\omega_1,\omega_2)} = \frac{G(z;p\omega_2,-\omega_1)}{G(0;p\omega_2,-\omega_1)}.$$

- These follow immediately from the lattice subset properties.

- The "$F(0;\cdots,\cdots)$" and "$G(0;\cdots,\cdots)$" can be removed by logarithmic differentiation. Then expand in powers of z and equate coefficients...

Generalized Eisenstein series

- $$\frac{x}{e^{px}-1}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)e^{kx}=\sum_{n=0}^{\infty}B_{n,p}\frac{x^n}{n!}$$

- $$E_n^0(\tau;\chi_p)=-\frac{B_{1,p}}{2}\delta_{n,1} + \sum_{j=1}^{\infty}\frac{j^{n-1}}{1-q^{pj}}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)q^{jk}$$

- $$E_n^{\infty}(\tau;\chi_p)=-\frac{B_{n,p}}{2n}+\sum_{j=1}^{\infty}\left(\frac{j}{p}\right)\frac{j^{n-1}q^j}{1-q^j}$$

- $$\delta_{m,n}=\begin{cases}1 & \text{if } m=n,\\ 0 & \text{if } m\neq n\end{cases}$$

Generalized Eisenstein series

Suppose p is an odd prime, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$, $n \in \mathbb{Z}^+$ and $n \equiv (p-1)/2 \pmod 2$. Then:

$$E_n^0\left(\frac{a\tau+b}{c\tau+d};\chi_p\right) = \left(\frac{d}{p}\right)(c\tau+d)^n E_n^0(\tau;\chi_p),$$

$$E_n^\infty\left(\frac{a\tau+b}{c\tau+d};\chi_p\right) = \left(\frac{d}{p}\right)(c\tau+d)^n E_n^\infty(\tau;\chi_p),$$

$$E_n^\infty\left(\frac{-1}{p\tau};\chi_p\right) = \frac{1}{c_p\sqrt{p}}(p\tau)^n E_n^0(\tau;\chi_p),$$

$$E_n^0\left(\frac{-1}{p\tau};\chi_p\right) = \frac{\sqrt{p}}{c_p}\tau^n E_n^\infty(\tau;\chi_p).$$

Remark

We could try and begin with

$$G_n^0(\tau;\chi_p) = \sum\sum{}' \frac{\left(\frac{k}{p}\right)}{(j+k\tau)^n}$$

and

$$G_n^\infty(\tau;\chi_p) = \sum\sum{}' \frac{\left(\frac{j}{p}\right)}{(j+pk\tau)^n}.$$

This requires $n > 2$ for convergence.

The method outlined in this talk yields results for $n = 1$ and $n = 2$ as well.

Part 5: Concluding remarks

- Venkatachaliengar's proof of the transformation formula for the eta-function is elementary and self-contained.

- It uses Liouville's theorem, but not the Jacobi triple product identity, or any other facts about theta functions.

- Venkatachaliengar's book contains many other elegant proofs. For example, the addition formula and differential equations for the Weierstrass and Jacobian elliptic functions are derived by simple (but clever) manipulations of series.

- Venkatachaliengar's question: can topics in the book be extended to finite fields?