EXPANDERS & RAMANUJAN GRAPHS

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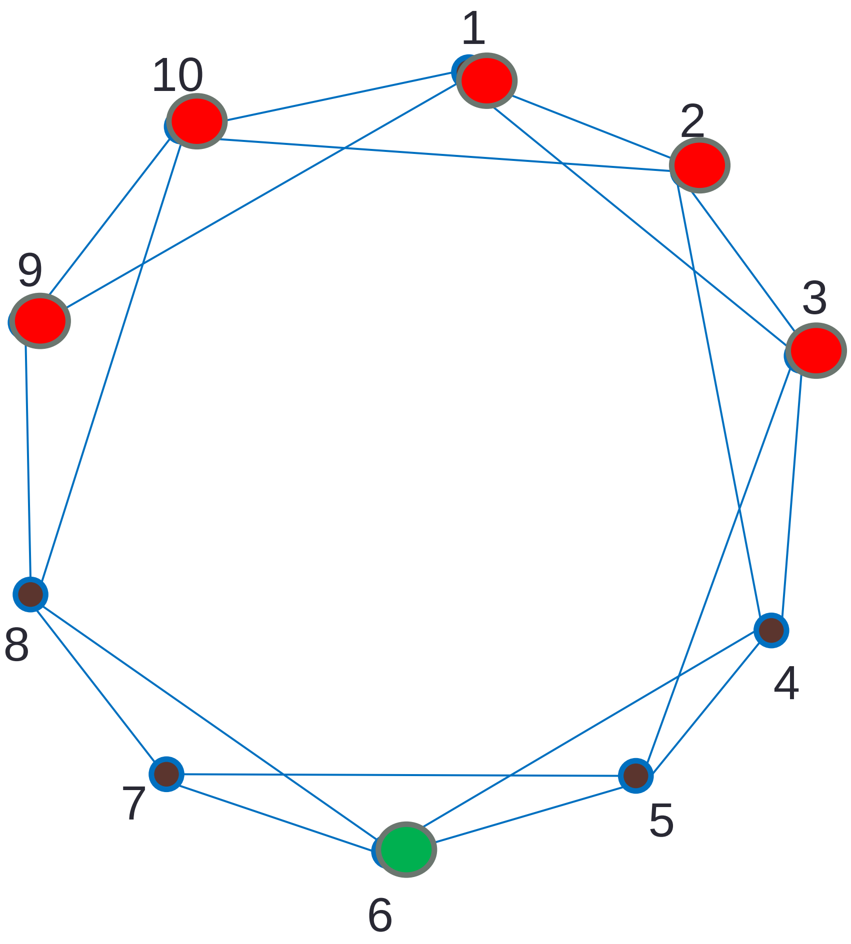
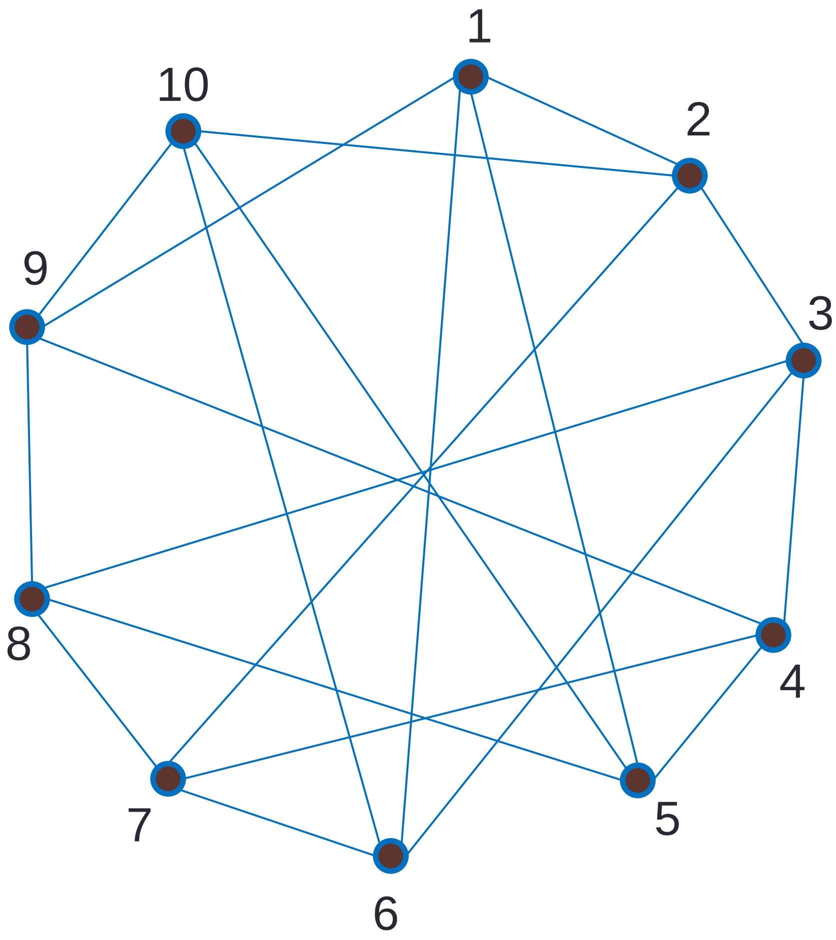
Highly Connected Graphs





Connectedness – Why is it important?

- **Efficient Communication Networks** – ‘well connected’ topologies achieve low latencies with as few links (cost?) as possible

- **Design of Rapidly Mixing Stochastic Processes** – a random walk on a well connected graph is likely to converge to its stationary distribution quickly

- **Pseudorandom Generators** – random walk on a well connected graph can be a very good source of pseudorandom bits – ‘randomness extraction’

- **Error Correcting Codes** – a ‘richly connected’ graph between messages and their codes (bipartite) is likely to lead to enough candidates among the codes that are mutually separated by a minimum distance (such as Hamming distance).

What is ‘Connectedness’?

- **Low ‘Diameter’** – largest of the minimum distances between pairs of nodes

$$\max_{u,v}\left(\min_{p_{u,v}} l(p_{uv})\right)$$

- **High ‘Expansion’** – any subset of nodes of the graph has ‘enough’ edges going out to those not in the subset

$$\min_{S\subseteq V, |S|\leq\frac{n}{2}} \frac{|\{E(u,v) \mid u \in S, v \in \overline{S}\}|}{|S|}$$

Isoperimetric Number of the graph

Spectral Graph Theory – Unifying Theme

- Deep relationships between the structural / combinatorial properties of a graph and the algebraic properties of its adjacency matrix. For a d-regular, n-vertex graph G:
  - Adjacency graph A is symmetric, each row/column adding up to d
  - $d = \lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq -d$ is the eigen-spectrum of A
  - G is Connected iff $\lambda_0 > \lambda_1$; multiplicity of λ_0 is the number of connected components of G
  - Maximum size of a clique in G is at most λ_1+1
  - G is Bipartite iff $\lambda_{n-1} = -d$; $\chi(G) \geq 1 - \frac{|\lambda_0|}{|\lambda_{n-1}|}$ (Chromatic Number)

(Hoffman, 1968)

  - Maximum size of a cut in G is at most $\left(\frac{|E|}{2} - ({}^{n}/_{4})\lambda_{n-1}\right)$

(Delorme & Poljak, 1993)

Spectral Gap and Connectivity

- $\lambda = (\lambda_0 - \lambda_1)$ is the **spectral gap** (λ_0 is in fact d)

- Diameter $\delta \leq \frac{\log(n-1)}{\log\left(d/\lambda_1\right)}$+1 (non-bipartite)

  $\delta \leq \frac{\log(n-2)/2}{\log\left(d/\lambda_1\right)}$+2 (bipartite)

  Small $\delta \Rightarrow$ small λ_1

  (Chung, 1989)

- Expansion Ratio h(G)

  $$\frac{\lambda}{2} \leq h(G) \leq \sqrt{2d\lambda}$$

  Large h(G) $\Rightarrow$ large λ $\Rightarrow$ *small* λ_1

  (Noga Alon & Milman, 1985)

Ramanujan Graphs

- $\lambda_1 \geq 2\sqrt{d-1}.\left(1 - o\left({}^{1}/_{\log^2 n}\right)\right)$

  (Noga Alon & Boppanna, 1991)

- For every integer d and ϵ, there exists a constant $\mathrm{c}(\epsilon, \mathrm{d})$ such that every (n, d)-graph G has at least $\mathrm{c.n}$ eigenvalues greater than $2\sqrt{d-1}$ - ϵ.

  (J-.P.Serre, 1991)

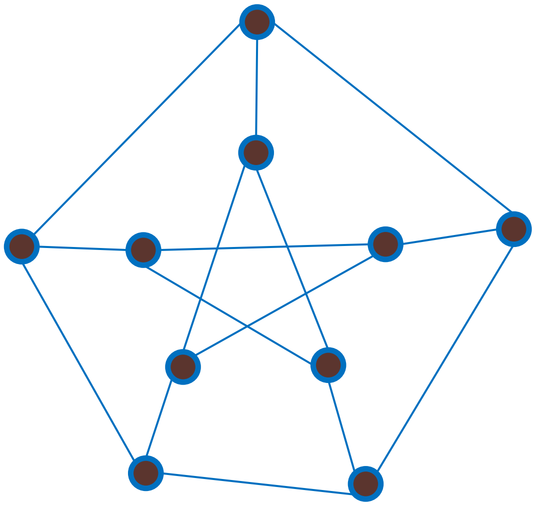
- d-regular Graphs with $\lambda_1 \leq 2\sqrt{d-1}$ are **Ramanujan Graphs**

  (Lubotzky, Phillips and Sarnak, 1988)

- Ramanujan Graphs have the largest spectral gap possible and therefore are the most 'well connected' (among d-regular graphs)

Examples of Ramanujan Graphs

- K_d, $K_{d,d}$ - are both Ramanujan
- Petersen Graph is Ramanujan



- Random d-regular graphs are 'almost' Ramanujan (Friedman 1991)

$$\lambda_1 \leq 2\sqrt{d-1} + 2\log d + O(1)$$

Expander Graphs

- Expander Graph Family: family of graphs $\boldsymbol{G_i}, \boldsymbol{i} \in \boldsymbol{N}$ such that:
  - $\boldsymbol{G_i}$ is a d-regular graph of size $\boldsymbol{n_i}$; $\{\boldsymbol{n_i}\}$ is a monotonically growing series that doesn't grow too fast (say $\boldsymbol{n_{i+1}} \ll \boldsymbol{n_i}^2$)
  - $\forall \boldsymbol{i}, \boldsymbol{h}(\boldsymbol{G_i}) \geq \boldsymbol{\epsilon} > \boldsymbol{0}$

- Example of an expander family (Super-Concentrators):
  - (n,m,d)-Superconcentrator is a bipartite graph with $|\mathrm{L}| = \mathrm{n}$, $|\mathrm{R}| = \mathrm{m}$ and every L-vertex has d neighbors.
  - Known (Pinsker 1973): a random superconcentrator satisfies the following with probability at least 0.9:

$$\text{for every } \boldsymbol{S} \subseteq \boldsymbol{L}, \boldsymbol{\Gamma}(\boldsymbol{S}) \geq \begin{cases} \frac{5d}{8}|\boldsymbol{S}|, & |\boldsymbol{S}| \leq \frac{n}{10d} \\ |\boldsymbol{S}|, & \frac{n}{10d} \leq |\boldsymbol{S}| \leq \frac{n}{2} \end{cases}$$

$\boldsymbol{\Gamma}(\boldsymbol{S})$ is the set of neighbors of S.

Properties of Expander Graphs

- Expander Families have $\boldsymbol{\delta = O(\log n)}$

- Expander Families are close to random: **Expander Mixing Lemma**: $\forall \boldsymbol{S}, \boldsymbol{T} \subseteq \boldsymbol{V}$,

$$\left| |\boldsymbol{E}(\boldsymbol{S}, \boldsymbol{T})| - \frac{\boldsymbol{d}|\boldsymbol{S}||\boldsymbol{T}|}{\boldsymbol{n}} \right| \leq \boldsymbol{\lambda_1} \sqrt{|\boldsymbol{S}||\boldsymbol{T}|}$$

Margulis Construction (1973)

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$$G(Z_m\, X\, Z_m) = (x, y) \rightarrow \left\{\begin{matrix} (x \pm y, y) \\ (x \pm (y+1), y) \\ (x, y \pm x) \\ \left(x, y \pm (x+1)\right) \end{matrix}\right\}, \text{ mod m}$$

Graphs with m^2 vertices and $\lambda_1 \leq 8$

Expander Families that are Ramanujan

- What we really need are families of expander graphs that are Ramanujan (Ramanujan Families)

- Constructions exist now – but are highly non-trivial. Proofs of ‘Ramanujan-ness’ have used a wide range of deep mathematics – Representation Theory of Lie Groups, Number Theory, Algebraic Geometry, …

Examples of Ramanujan Families

- Lubotzky, Phillips and Sarnak (1988): $\boldsymbol{V_p} = \boldsymbol{Z_p}$ for some prime p, $\boldsymbol{d = 3}$, x is connected to x+1, x-1 and $\boldsymbol{x^{-1}}$ modulo p.

- The proof crucially depended on the **Ramanujan-Petersson Conjecture** (now a theorem): that the Ramanujan Tau function,

$$\sum_{n\geq 1} \boldsymbol{\tau(n) q^n} = \boldsymbol{q} \prod_n (\boldsymbol{1} - \boldsymbol{q^n})^{24}, \text{ where } \boldsymbol{q} = \boldsymbol{e^{2\pi i z}}$$

satisfies: $|\boldsymbol{\tau(p)}| \leq \boldsymbol{2p^{\frac{11}{2}}}$,for all primes p.

Hence the name **Ramanujan Graphs**.

Ramanujan Families of Arbitrary size?

- Construction of Ramanujan families for any $\mathbf{n}$ other than primes and prime powers remains an important open problem.

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THANK YOU

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