MODULAR EQUATIONS IN THE SPIRIT OF RAMANUJAN

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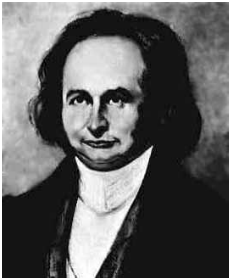
The theory of modular equations dates back to 1771 and 1775 when J. Landen records Landen's transformation in his papers

- J. Landen, A disquisition concerning certain fluents, which are assignable by the arcs of the conic sections; where in are investigated some new and useful theorems for computing such fluents, Philos. Trans. R. Soc., London, 61 (1771), 298-309.
- J. Landen, An investigation of a general theorem for finding the length of any arc of any conic hyperbola, by means of two elliptic arcs, with some other new and useful theorems deduced therefrom, Philos. Trans. R. Soc., London 65 (1775), 283-289.

A. M. Legendre

But actually the theory commenced when A. M. Legendre [Traite..des fonctions elliptiques, t. 1, Huzard-Coureier, Paris, 1825] derived a modular equation of degree 3 in 1825.



Carl Gustav Jakob Jacobi (1804 - 1851)

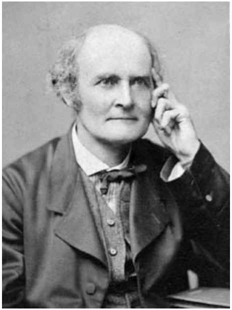
C. G. J. Jacobi established modular equations of degrees 3 and 5 in his famous book Fundamenta Nova in 1829.

Subsequently many mathematicians have contributed to the theory of modular equations. Some of them are C. Guetzlaff, L. A. Sohncke, E. Fiedler, R. Fricke, R. Russell.

L. Schläfli (1814 - 1895)

H. E. Schröter (1829 - 1892)

F. Klein (1849 - 1925)



A. Cayley (1821 - 1895)

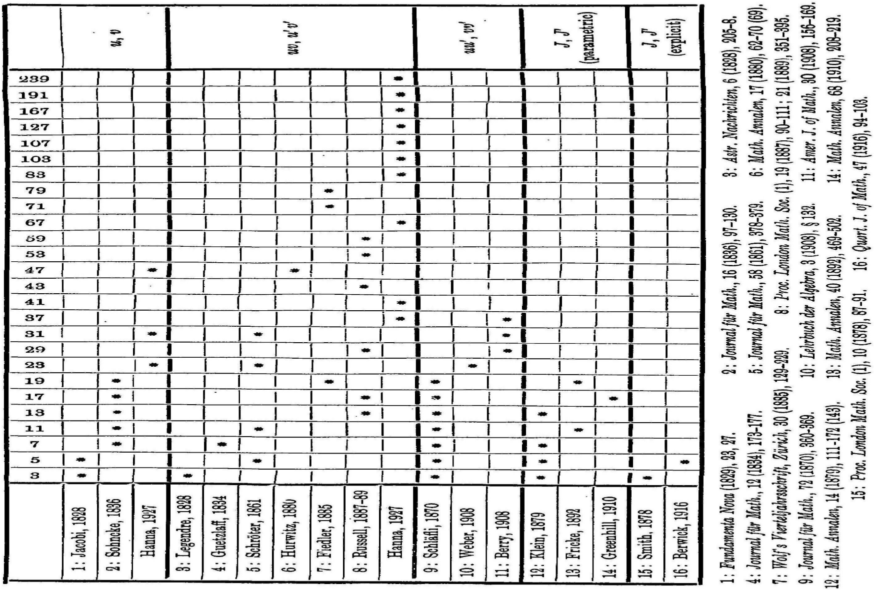
A. Hurwitz (1859 - 1919)

H. Weber (1842 - 1913)

H. Weber derived several mixed Schläfli-type modular equations.

- H. Weber, Zur Theorie der elliptischen Functionen, Acta Math., 11 (1887), 333–390.

More information about modular equations can be found in the books of A. Enneper, Weber, Klein and Fricke.



		3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	67	71	79	83	103	107	127	167	191	239
u, v	1: Jacobi, 1828	*	*																								
	2: Sohncke, 1836			*	*	*	*	*																			
	Hanna, 1927								*		*				*												
$uv, u'v'$	3: Legendre, 1828	*																									
	4: Guetzlaff, 1834			*																							
	5: Schröter, 1861		*		*				*		*																
	6: Hurwitz, 1880														*												
	7: Fiedler, 1885							*											*	*							
	8: Russell, 1887–89					*	*			*				*		*	*										
	Hanna, 1927											*	*					*			*	*	*	*	*	*	*
uu', vv'	9: Schläfli, 1870	*	*	*	*	*	*	*																			
	10: Weber, 1908								*																		
	11: Berry, 1908									*	*	*															
J, J' (parametric)	12: Klein, 1879	*	*	*		*																					
	13: Fricke, 1892				*			*																			
	14: Greenhill, 1910						*																				
J, J' (explicit)	15: Smith, 1878	*																									
	16: Berwick, 1916		*																								

1: *Fundamenta Nova* (1829), 23, 27. 2: *Journal für Math.*, 16 (1836), 97–130. 3: *Astr. Nachrichten*, 6 (1828), 205–8.
4: *Journal für Math.*, 12 (1834), 173–177. 5: *Journal für Math.*, 58 (1861), 378–379. 6: *Math. Annalen*, 17 (1880), 62–70 (69).
7: *Wolf's Vierteljahrsschrift, Zürich*, 30 (1885), 129–229. 8: *Proc. London Math. Soc.* (1), 19 (1887), 90–111; 21 (1889), 351–395.
9: *Journal für Math.*, 72 (1870), 360–369. 10: *Lehrbuch der Algebra*, 3 (1908), § 132. 11: *Amer. J. of Math.*, 30 (1908), 156–169.
12: *Math. Annalen*, 14 (1879), 111–172 (143). 13: *Math. Annalen*, 40 (1892), 469–502. 14: *Math. Annalen*, 68 (1910), 208–219.
15: *Proc. London Math. Soc.* (1), 10 (1878), 87–91. 16: *Quart. J. of Math.*, 47 (1916), 94–103.

Modular equation by Hanna, Proc. london math. soc., 1928.

Srinivasa Ramanujan (1887 - 1920)

But Ramanujan's contributions in this area of modular equations are immense. Perhaps Ramanujan found more modular equations than all of his predecessors discovered together. Ramanujan has recorded many modular equations in his notebooks, which are very useful in the computation of class invariants and the values of theta-functions.

Ramanujan begins his study of “modular equations” in Chapter 15 by defining

$$F(x) := (1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!} x^n = {}_1F_0\left(\frac{1}{2}; x\right), \quad |x| < 1.$$

He then states the trivial identity

$$F\left(\frac{2t}{1+t}\right) = (1+t)F\left(t^2\right). \qquad (*)$$

After setting $\alpha = 2t/(1+t)$ and $\beta = t^2$, Ramanujan offers the “modular equation of degree 2”,

$$\beta(2-\alpha)^2 = \alpha^2,$$

which is readily verified. The factor $(1+t)$ in (*) is called the multiplier. He then derives some modular equations of higher degree and offers some general remarks.

Ramanujan's theta-function

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \tag{1}$$

as defined by Srinivasa Ramanujan in Chapter 16, of his Second notebook.

By using Jacobi's triple product identity, product representation of $f(a,b)$ is

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{2}$$

where

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}), \quad |q|<1.$$

Special cases of $f(a,b)$

$$\varphi(q) := f(q,q) = \sum_{n=0}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{4}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=0}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{5}$$

$$\chi(q) := (-q;q^2)_\infty, \tag{6}$$

Let

$$Z(r) := Z(r;x) := {}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right) \tag{7}$$

and

$$q_r := q_r(x) := \exp\left(-\pi \csc\left(\frac{\pi}{r}\right) \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)}\right), \tag{8}$$

where $r = 2, 3, 4, 6$ and $0 < x < 1$.

If n is a fixed positive integer, then the modular equation of degree n in the theory of signature r is a relation between α and β induced by the equation

$$n \ \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha)} = \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \beta)}. \tag{9}$$

The multiplier $m(r)$ is defined by

$$m(r) := \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \beta)}. \tag{10}$$

where $r = 2, 3, 4, 6$.
When the contest is clear, we omit the argument r in q_r, Z_r and $m(r)$.

Definition

Ramanujan class invariant G_n is defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) = (4\alpha(1-\alpha))^{-1/24},$$
$$g_n := 2^{-1/4}q^{-1/24}\chi(-q) = (4\alpha(1-\alpha)^{-2})^{-1/24}, \quad (11)$$

where

$$\chi(q) = (-q;q^2)_\infty, \quad q = exp(-\pi\sqrt{n})$$

and

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n) \quad |q| < 1.$$

If $P = \frac{f^2(-x)}{x^{\frac{1}{6}} f^2(-x^3)}$ and $Q = \frac{f^2(-x^4)}{x^{\frac{1}{3}} f^2(-x^6)}$, then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3.$$

If $P = \frac{f(-x^2)}{x^{\frac{1}{24}} f(-x^3)}$ and $Q = \frac{f(-x)}{x^{\frac{5}{24}} f(-x^6)}$, then

$$(PQ)^2 - \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - 8\cdot\left(\frac{P}{Q}\right)^3.$$

If $P = \frac{f(-x^2)}{x^{\frac{1}{8}} f(-x^5)}$ and $Q = \frac{f(-x)}{x^{\frac{3}{8}} f(-x^{10})}$, then

$$PQ - \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 - 4\cdot\left(\frac{P}{Q}\right)^2.$$

If $P = \frac{f^2(-x)}{x^{\frac{1}{2}} f^2(-x^7)}$ and $Q = \frac{f^2(-x^4)}{x^{\frac{3}{2}} f^2(-x^{14})}$, then

$$PQ + \frac{49}{PQ} = \left(\frac{Q}{P}\right)^3 - 8\cdot\frac{Q}{P} - 8\cdot\frac{P}{Q} + \left(\frac{P}{Q}\right)^3.$$

If $P = \frac{f(-x)}{x^{\frac{1}{2}} f(-x^{13})}$ and $Q = \frac{f(-x^2)}{x f(-x^{26})}$, then

$$PQ + \frac{13}{PQ} = \left(\frac{Q}{P}\right)^3 - 4\cdot\frac{Q}{P} - 4\cdot\frac{P}{Q} + \left(\frac{P}{Q}\right)^3.$$

If $P = \frac{f(-x)}{x^{\frac{1}{3}} f(-x^9)}$ and $Q = \frac{f(-x^4)}{x^{\frac{2}{3}} f(-x^{18})}$, then

$$P^3 + Q^3 = P^2Q^2 + 3PQ.$$

If $P = \frac{\psi(x)}{x^{\frac{1}{2}} \psi(x^5)}$ and $Q = \frac{\psi(x^3)}{x^{\frac{3}{2}} \psi(x^{15})}$, then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 + 3\cdot\frac{Q}{P} + 3\cdot\frac{P}{Q} - \left(\frac{P}{Q}\right)^2.$$

If $P = \frac{\phi(x)}{\phi(x^5)}$ and $Q = \frac{\phi(x^3)}{\phi(x^{15})}$, then

$$5PQ + \frac{1}{PQ} = \left(\frac{P}{Q}\right)^2 + 3\cdot\frac{P}{Q} + 3\frac{Q}{P} - \left(\frac{Q}{P}\right)^2.$$

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$$u = \frac{f(-x)}{\sqrt[5]{x}\,f(-x^5)}\ ; \quad v = \frac{f(-x^n)}{\sqrt[5]{x^n}\,f(-x^{5n})}$$

$n = 2$. $$uv + \frac{5}{uv} = \left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3$$

$n = 3$. $$(uv)^3 + \left(\frac{5}{uv}\right)^3 = -\left\{\left(\frac{u}{v}\right)^6 - \left(\frac{v}{u}\right)^6\right\} - 9\left\{\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3\right\}.$$

$n = 4$. $$(uv)^3 + \left(\frac{5}{uv}\right)^3 = \left(\frac{u}{v}\right)^5 + \left(\frac{v}{u}\right)^5 - 8\left\{\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3\right\} + 4\left(\frac{u}{v} + \frac{v}{u}\right) + \frac{4}{\frac{u}{v} + \frac{v}{u}}.$$

$n = 5$. $$(uv)^2 + \left(\frac{5}{uv}\right)^2 + 5\left(uv + \frac{5}{uv}\right) + 15 = \left(\frac{v}{u}\right)^3.$$

$n = 7$. $$(uv)^3 + \left(\frac{5}{uv}\right)^3 = -\left\{\left(\frac{u}{v}\right)^4 - \left(\frac{v}{u}\right)^4\right\} - 7\left\{\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3\right\} - 7\left\{\left(\frac{u}{v}\right)^2 - \left(\frac{v}{u}\right)^2\right\} + 14\left(\frac{u}{v} + \frac{v}{u}\right).$$

$$u^2 = \frac{f(-x^{1/5})}{\sqrt[5]{x}\,f(-x^5)}\ ; \quad v^2 = \frac{f(-x^{n/5})}{\sqrt[5]{x^n}\,f(-x^{5n})}$$

$n = 2$ $$uv + \frac{5}{uv} = \left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 - 2\left\{\frac{u}{v} + \frac{v}{u}\right\}$$

$n = 3$ $$(uv)^2 + \left(\frac{5}{uv}\right)^2 + 3\left(\frac{u}{v} + \frac{v}{u}\right)\left(uv + \frac{5}{uv}\right) = \left(\frac{u}{v}\right)^2 + \left(\frac{v}{u}\right)^2 - 6\left(\frac{u}{v} + \frac{v}{u}\right) - 9.$$

~~$n = 4$~~

$$u = 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \cdots$$

$$v = 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \cdots$$

$$u^{11} v = 1 + 11x u^6 v^6 + x^2 u v^{11}.$$

$$u = \frac{f(-x)}{x^{\frac{n-1}{24}} f(-x^n)} \qquad v = \frac{f(-x^5)}{x^{\frac{5(n-1)}{24}} f(-x^{5n})}$$

$n = 2$. $$(uv)^2 + \left(\frac{2}{uv}\right)^2 = \left(\frac{v}{u}\right)^3 - \left(\frac{u}{v}\right)^3.$$

$n = 3$. $$(uv)^2 + \left(\frac{3}{uv}\right)^2 + 5 = \left(\frac{v}{u}\right)^3 - \left(\frac{u}{v}\right)^3.$$

$n = 4$. $$(uv)^2 + \left(\frac{4}{uv}\right)^2 = \left(\frac{v}{u}\right)^3 + \left(\frac{u}{v}\right)^3 - 5\left\{\frac{v}{u} + \frac{u}{v}\right\}.$$

$n = 5$. $$(uv)^2 + \left(\frac{5}{uv}\right)^2 + 5\left\{uv + \frac{5}{uv}\right\} + 15 = \left(\frac{v}{u}\right)^3$$

$n = 7$. $$(uv)^2 + \left(\frac{7}{uv}\right)^2 - 5 = \left(\frac{v}{u}\right)^3 - \left(\frac{u}{v}\right)^3 - 5\left\{\left(\frac{v}{u}\right)^2 + \left(\frac{u}{v}\right)^2\right\}$$

~~$n = 9$.~~

$$u = \frac{f(-x^n)}{x^{\frac{5-n}{24}} f(-x^5)}\ ; \quad v = \frac{f(-x)}{x^{\frac{5n-1}{24}} f(-x^{5n})}$$

$n = 2$. $$uv - \frac{5}{uv} = \left(\frac{v}{u}\right)^2 - \left(\frac{2u}{v}\right)^2$$

$n = 3$. $$(uv)^3 - \left(\frac{5}{uv}\right)^3 = \left(\frac{v}{u}\right)^4 - \left(\frac{3u}{v}\right)^4 + \left(\frac{v}{u}\right)^2 - \left(\frac{3u}{v}\right)^2$$

Ramanujan-type P–Q eta-function identities

In Chapter 25 of his Second Notebook, Ramanujan states twenty three beautiful $P - Q$ eta-function identities. These are identities involving quotients of eta-functions, which are designated by P or Q by Ramanujan. Elementary proofs of eighteen of these twenty three P - Q identities have been proved by B. C. Berndt and L.-C. Zhang in

- B. C. Berndt and L.-C. Zhang, *Ramanujan's identities for eta-functions*, Math. Ann., 292 (1992), 561-573,

on employing various modular equations of Ramanujan. To establish the remaining five $P - Q$ eta-function identities Berndt has used the theory of modular forms. The modular equations are very important to derive $P - Q$ eta-function identities.

In

- M. S. Mahadeva Naika, $P-Q$ *eta-function identities and computation of Ramanujan-Weber class invariants*, J. Indian Math. Soc., 70(1-4), (2003), 121–134,

has established several $P-Q$ eta-function identities. Proof consists in elementary algebraic manipulation of modular equation in the theory of signature 2 and 4.

Theorem (M. S. Mahadeva Naika, J. Indian Math. Soc., 2003)

If

$$P = \frac{f(-q)}{q^{1/24} f(-q^2)} \qquad and \qquad Q_n = \frac{f(-q^n)}{q^{n/24} f(-q^{2n})}, \tag{12}$$

then

$$(PQ_3)^3 + \frac{8}{(PQ_3)^3} = \left(\frac{Q_3}{P}\right)^6 - \left(\frac{P}{Q_3}\right)^6, \tag{13}$$

Proof. Let

$$b(q) = \frac{f^2(-q)}{f(-q^2)} \qquad and \qquad c(q) = \frac{2^{3/4} q^{1/8} f^2(-q^2)}{f(-q)}. \tag{14}$$

Using Theorems 9.9 and 9.10 of Ch.33, p.148 (Ramanujan's Notebook Part-V) in the modular equation of degree 3 in signature 4, we find that

$$b^4(q)b^4(q^3) + c^4(q)c^4(q^3) + 4b^2(q)b^2(q^3)c^2(q)c^2(q^3) = Z_1^2 Z_3^2,$$

which is equivalent to

$$\frac{f^6(-q)f^6(-q^3)}{qf^6(-q^2)f^6(-q^6)} + \frac{64qf^6(-q^2)f^6(-q^6)}{f^6(-q)f^6(-q^3)} + 32$$

$$= \left[\frac{f^{12}(-q)}{q^{1/2}f^{12}(-q^2)} + \frac{64q^{1/2}f^{12}(-q^2)}{f^{12}(-q)}\right]^{1/2}\left[\frac{f^{12}(-q^3)}{q^{3/2}f^{12}(-q^6)} + \frac{64q^{3/2}f^6(-q^6)}{f^6(-q^3)}\right]^{1/2}.$$

Using (12) with $n = 3$ in the above identity, we find that

$$(PQ_3)^6 + \frac{64}{(PQ_3)^6} + 32 = \left(P^{12} + \frac{64}{P^{12}}\right)^{1/2}\left(Q_3^{12} + \frac{64}{Q_3^{12}}\right)^{1/2}.$$

Squaring both sides of the above identity and after some simplification, we obtain the required result.

If P and Q_n are defined as in (12), then

$$(PQ_5)^2 + \frac{4}{(PQ_5)^2} = \left(\frac{Q_5}{P}\right)^3 - \left(\frac{P}{Q_5}\right)^3, \tag{15}$$

$$\begin{aligned}(PQ_7)^9 + \frac{512}{(PQ_7)^9} + 21\left((PQ_7)^6 + \frac{64}{(PQ_7)^6}\right)\\ + 168\left((PQ_7)^3 + \frac{8}{(PQ_7)^3}\right) + 658 = \left(\frac{Q_7}{P}\right)^{12} + \left(\frac{P}{Q_7}\right)^{12},\end{aligned} \tag{16}$$

$$\begin{aligned}(PQ_{11})^{10} + \frac{1024}{(PQ_{11})^{10}} + 22\left((PQ_{11})^8 + \frac{256}{(PQ_{11})^8}\right)\\ + 209\left((PQ_{11})^6 + \frac{64}{(PQ_{11})^6}\right) + 1144\left((PQ_{11})^4 + \frac{16}{(PQ_{11})^4}\right)\\ + 4048\left((PQ_{11})^2 + \frac{4}{(PQ_{11})^2}\right) + 9744 = \left(\frac{Q_{11}}{P}\right)^{12} + \left(\frac{P}{Q_{11}}\right)^{12}.\end{aligned} \tag{17}$$

Theorem (M. S. Mahadeva Naika, J. Indian Math. Soc., 2003)

If

$$P = \frac{f(-q)}{q^{1/8}f(-q^4)} \qquad and \qquad Q_n = \frac{f(-q^n)}{q^{n/8}f(-q^{4n})}, \tag{18}$$

then

$$PQ_3 + \frac{4}{PQ_3} = \left(\frac{Q_3}{P}\right)^2 + \left(\frac{P}{Q_3}\right)^2, \tag{19}$$

Proof. Let

$$B(q) = \frac{f^{8/5}(-q)}{f^{2/5}(-q^4)} \qquad and \qquad C(q) = \frac{2q^{1/4}f^{8/5}(-q^4)}{f^{2/5}(-q)}. \tag{20}$$

Using Entry 12(ii), (iv), of Ch.17, pp.123-124 (Ramanujan's Notebooks Part-III by Berndt) in the modular equation of degree 3, we find that

$$B(q)B(q^3) + C(q)C(q^3) = Z_1^{3/5}Z_3^{3/5},$$

which is equivalent to

$$\frac{4q^{1/2}f(-q^4)f(-q^{12})}{f(-q)f(-q^3)}+\frac{f(-q)f(-q^3)}{q^{1/2}f(-q^4)f(-q^{12})}$$

$$=\left[\frac{f^4(-q)}{q^{1/2}f^4(-q^4)}+\frac{16q^{1/2}f^4(-q^4)}{f^4(-q)}\right]^{1/4}\left[\frac{f^4(-q^3)}{q^{3/2}f^4(-q^{12})}+\frac{16q^{3/2}f^4(-q^{12})}{f^4(-q^3)}\right]^{1/4}.$$

Using (18) with $n=3$ in the above identity, we deduce that

$$\frac{4}{PQ_3}+PQ_3=\left(P^4+\frac{16}{P^4}\right)^{1/4}\left(Q_3^4+\frac{16}{Q_3^4}\right)^{1/4}.$$

After some simplification of the above identity, we obtain the required result. If P and Q_n and defined as in (18), then

$$\begin{aligned}&\left((PQ_7)^3+\frac{64}{(PQ_7)^3}\right)+7\left((PQ_7)^2+\frac{16}{(PQ_7)^2}\right)\\&+28\left(PQ_7+\frac{4}{PQ_7}\right)+70=\left(\frac{Q_7}{P}\right)^4+\left(\frac{P}{Q_7}\right)^4.\end{aligned}\tag{21}$$

In the following theorem, we establish five P-Q eta-function identities with 4 moduli. In what follows $Im(z) > 0$, and $Q := Q(z) = P(2z)$ always.

Theorem

We have

$$Q^2 - P^2Q - 4P = 0, \ where\ P := P(z) = \frac{\eta^2(z)\eta^2(3z)}{\eta^2(2z)\eta^2(6z)}, \tag{22}$$

$$Q^2 - (P^2 + \sqrt[3]{4}P)Q - 16P = 0, \ where\ P := P(z) = \frac{\eta^4(z)\eta^4(5z)}{\eta^4(2z)\eta^4(10z)}, \tag{23}$$

$$Q^2 - P^2Q - 2P = 0, \ where\ P := P(z) = \frac{\eta(z)\eta(7z)}{\eta(2z)\eta(14z)}, \tag{24}$$

$$Q^2 - (8P + P^2)Q - 4P = 0, \ where\ P := P(z) = \frac{\eta^2(z)\eta^2(11z)}{\eta^2(2z)\eta^2(22z)}, \tag{25}$$

$$Q^2 - (P^2 + 2P)Q - 2P = 0, \ where\ P := P(z) = \frac{\eta(z)\eta(23z)}{\eta(2z)\eta^2(46z)}. \tag{26}$$

In

- S. Bhargava, C. Adiga and M. S. Mahadeva Naika, *A new class of modular equations akin to Ramanujan's P-Q eta-function identities and some evaluations there from*, Adv. Stud. Contemp. Math., 5 (2002), No. 1, 37-48,

the authors have obtained new P-Q identities similar to those of Ramanujan which are as follows: In what follows $Im\ z > 0$ and $Q := Q(z) := P(2z)$ always.

(i)

$$P^2Q^2 + 3PQ = Q^3 + P^3 - 2(P^2Q + PQ^2),$$
$$\text{where } P := P(z) := \frac{\eta(z)\eta(11z)}{\eta(3z)\eta(33z)},\ \eta(z) = q^{1/24}f(-q), \tag{27}$$

(ii)

$$PQ + \frac{1}{PQ} = \left[\frac{P}{Q}\right]^3 + \left[\frac{Q}{P}\right]^3 + 4\left[\frac{P}{Q} + \frac{Q}{P}\right],$$
$$\text{where } P := P(z) := \frac{\eta(z)\eta(21z)}{\eta(3z)\eta(7z)}, \tag{28}$$

(iii)

$$P^2Q^2 + 7PQ = Q^3 + P^3 - 2(P^2Q + PQ^2),$$
$$\text{where } P := P(z) := \frac{\eta(z)\eta(3z)}{\eta(7z)\eta(21z)}, \tag{29}$$

(iv)

$$P^2Q^2 + PQ = Q^3 + P^3 + 2(P^2Q + PQ^2),$$
$$\text{where } P := P(z) := \frac{\eta(13z)\eta(3z)}{\eta(39z)\eta(z)}, \tag{30}$$

(v)

$$P^2Q^2 - PQ = Q^3 + P^3 + 2(P^2Q + PQ^2),$$
$$\text{where } P := P(z) := \frac{\eta(7z)\eta(5z)}{\eta(35z)\eta(z)}, \tag{31}$$

(vi)

$$\frac{1}{PQ} - 3PQ = \left[\frac{Q}{P}\right]^3 + \left[\frac{P}{Q}\right]^3, \qquad (32)$$

$$\text{where } P := P(z) := \frac{\eta(z)\eta(9z)}{\eta(3z)\eta(3z)},$$

Modular equations of degree 2

In

- M. S. Mahadeva Naika and S. Chandankumar, Some new modular equations and their applications, Tamsui Oxf. J. Math. Sci., (accepted).
- M. S. Mahadeva Naika, S. Chandankumar and M. Manjunatha, On some new modular equations and their applications to continued fractions, Int. Math. Forum, 6 (58), 2011, 2881–2905.
- M. S. Mahadeva Naika, K. Sushan Bairy and Chandankumar S, Some new modular equations of degree 2 akin to Ramanujan, (communicated).

we have established several modular equations of degree 2 and as an application we have obtained some new modular relations and explicit evaluations of Ramanujan-Göllnitz-Gordon Continued Fraction, Ramanujan-Selberg continued fraction and continued fraction of Eisenstein.

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^4)}{\varphi(q^8)}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^8)}{\varphi(q^4)}$, *then*

$$4Q^2 + \frac{1}{Q^2} - 4\left(2Q + \frac{3}{Q}\right) + 4\left(P + \frac{2}{P}\right) + \left(P^2 + \frac{4}{P^2}\right) + 16 = 4\left(PQ + \frac{1}{PQ}\right) + 2\left(\frac{P}{Q} + \frac{4Q}{P}\right). \quad (33)$$

Proof of (33)**:** If β is of degree 4 over α [Ch. 18, Vol. 2, Entry 24(v), pp. 228], then

$$\beta = \left\{\frac{1-\sqrt[4]{1-\alpha}}{1+\sqrt[4]{1-\alpha}}\right\}^4. \quad (34)$$

From Entry 10 (i), (iv) of Ramanujan's second notebook, pp. 210, for $0 < x < 1$, we have

$$\varphi(e^{-y}) = \sqrt{z}. \quad (35)$$

$$\varphi(e^{-2y}) = \sqrt{z}\left(\frac{1}{2}(1+\sqrt{1-x})\right)^{1/2}, \quad (36)$$

where $z = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$ and $y = \pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}$.

Using the equations (35) and (36) in the equation (84), we deduce that

$$\begin{aligned} &-8Q^3P-2P^3Q-P^2-4Q^2+4QP+8Q^2P^2+2P^3cQ \\ &-8Q^3cP+8Q^2P^2c+4Q^4P^2-4Q^3P^3+P^4Q^2=0. \end{aligned} \tag{37}$$

where $c = \sqrt[4]{1-\alpha}$.

Isolating the terms having c on one side of the equation (37) and squaring both sides, we find that

$$\begin{aligned} &(P^4Q^2+P^2-2P^3Q-12QP^2-4QP+4Q^2P^3+16Q^2P^2 \\ &+8Q^2P+4Q^2-4Q^3P^3-8Q^3P^2-8Q^3P+4Q^4P^2)(P^4Q^2 \\ &+P^2-2P^3Q+12QP^2-4QP-4Q^2P^3+16Q^2P^2-8Q^2P \\ &+4Q^2-4Q^3P^3+8Q^3P^2-8Q^3P+4Q^4P^2)=0. \end{aligned} \tag{38}$$

By examining the behaviour of the second factor near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the second factor is not zero, whereas the first factor is zero in this neighbourhood. By the Identity Theorem first factor vanishes identically. Hence, we obtain the equation (33).

Theorem

If $P := \dfrac{\varphi(q)}{\varphi(q^2)}$ *and* $Q := \dfrac{\varphi(q^{3/2})}{\varphi(q^3)}$, *then*

$$\begin{aligned}
&288P^2Q^2 - 384P^2Q^4 - 32P^2 + 24P^4 + 584Q^4P^4 - 432Q^2P^4 - 216P^6Q^4 \\
&+ 152P^6Q^2 + 6P^8Q^4 - 4P^8Q^2 + Q^8P^8 - 4Q^6P^8 + 72Q^6P^6 + 80P^3Q + 16 \\
&- 96PQ - 48Q^3P^3 - 8P^6 + P^8 - 192P^4Q^6 + 128P^2Q^6 + 64Q^3P + 16Q^7P^5 \\
&- 24Q^5P^5 - 48Q^3P^5 + 56QP^5 - 12Q^7P^7 + 20Q^5P^7 + 28Q^3P^7 - 36QP^7 = 0.
\end{aligned} \tag{39}$$

Theorem

If $P := \dfrac{\varphi(q)}{\varphi(q^2)}$ *and* $Q := \dfrac{\varphi(q^{5/2})}{\varphi(q^5)}$, *then*

$$
\begin{aligned}
&64 + 128560P^4Q^4 - 31840P^4Q^2 - 3980P^{10}Q^8 + 9200P^{10}Q^6 - 63280P^8Q^6 - 192P^2 \\
&+ 32140P^8Q^8 + 161536P^6Q^6 - 126560P^6Q^4 + 36800P^6Q^2 - 160P^6 + 40Q^{11}P^{11} \\
&+ 240P^4 + 12800P^2Q^2 + 60920P^8Q^4 - 11720P^{10}Q^4 - 23440P^8Q^2 + 5712P^{10}Q^2 \\
&+ 15P^{12}Q^4 - 6P^{12}Q^2 + P^{12}Q^{12} + 15P^{12}Q^8 - 20P^{12}Q^6 - 6P^{12}Q^{10} + P^{12} + 60P^8 \\
&- 12P^{10} - 6400P^8Q^{10} - 89600P^6Q^8 + 102400P^4Q^8 + 2560Q^7P^3 - 7680Q^5P^3 \\
&+ 5760Q^3P^3 - 640QP^3 - 51200P^2Q^4 + 17920P^6Q^{10} - 20480P^4Q^{10} + 800P^{10}Q^{10} \\
&- 40960Q^8P^2 + 8192Q^{10}P^2 + 71680Q^6P^2 + 1024Q^5P - 2560Q^3P + 1280QP \\
&- 11520Q^7P^5 + 1280Q^9P^5 + 22400Q^5P^5 - 6400Q^3P^5 - 5504QP^5 + 11200Q^7P^7 \\
&- 1920Q^9P^7 - 18880Q^5P^7 + 128Q^{11}P^7 + 2880Q^3P^7 + 6592QP^7 - 1600Q^7P^9 \\
&+ 720Q^9P^9 + 1440Q^5P^9 - 160Q^{11}P^9 + 1120Q^3P^9 - 1520QP^9 - 688Q^7P^{11} \\
&- 40Q^9P^{11} + 1648Q^5P^{11} - 760Q^3P^{11} - 200QP^{11} - 179200P^4Q^6 = 0.
\end{aligned}
\tag{40}
$$

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^2)} \dfrac{\varphi(q^5)}{\varphi(q^{10})}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^2)} \dfrac{\varphi(q^{10})}{\varphi(q^5)}$, *then*

$$
\begin{aligned}
& Q^3 - \frac{1}{Q^3} + 20\left(Q^2 + \frac{1}{Q^2}\right) + 5\left(Q - \frac{1}{Q}\right) + 16\left(P^2 + \frac{4}{P^2}\right) \\
& + 120 = 40\left(Q + \frac{1}{Q}\right)\left(P + \frac{2}{P}\right).
\end{aligned} \tag{41}
$$

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^2)}$ *and* $Q = \dfrac{\varphi(q^6)}{\varphi(q^{12})}$, *then*

$$\begin{aligned}
&Q^8(P^8+1) + 4(-4Q^5+3Q^7)P^7 + 4(-Q^8-48Q^4+32Q^2+18Q^6)P^6 + 16 \\
&+ 2(292Q^4+3Q^8-108Q^6-192Q^2)P^4 + 4(12Q^3-16Q-7Q^7+12Q^5)P^3 \\
&+ 4(-108Q^4-Q^8+38Q^6+72Q^2)P^2 + 4(-14Q^5+9Q^7+24Q-20Q^3)P \\
&+ 4(-5Q^7+6Q^5)P^5 - 32Q^2 + 24Q^4 - 8Q^6 = 0.
\end{aligned} \tag{42}$$

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^7)}{\varphi(q^{14})}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^{14})}{\varphi(q^7)}$, *then*

$$\begin{aligned} &Q^4 + \frac{1}{Q^4} + 56\left(Q^3 + \frac{1}{Q^3}\right) + 252\left(Q^2 + \frac{1}{Q^2}\right) + 1064\left(Q + \frac{1}{Q}\right) \\ &- 64\left(P^3 + \frac{8}{P^3}\right) + 1078 = 112\left\{\left(P + \frac{2}{P}\right)\left[2\left(Q^2 + \frac{1}{Q^2}\right)\right.\right. \\ &\left.\left.+3\left(Q + \frac{1}{Q}\right) + 8\right] - \left(P^2 + \frac{4}{P^2}\right)\left[2\left(Q + \frac{1}{Q}\right) + 1\right]\right\}. \end{aligned} \tag{43}$$

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^8)}{\varphi(q^{16})}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^2)}\dfrac{\varphi(q^{16})}{\varphi(q^8)}$, *then*

$$
\begin{aligned}
&16Q^4 + \frac{1}{Q^4} + 8\left(8Q^3 - \frac{11}{Q^3}\right) + 64\left(4Q^2 + \frac{13}{Q^2}\right) + 16\left(74Q + \frac{113}{Q}\right) \\
&+ \left(P^4 + \frac{16}{P^4}\right) - 4\left(P^3 + \frac{8}{P^3}\right)\left[\left(2Q + \frac{1}{Q}\right) + 6\right] + 2\left(P^2 + \frac{4}{P^2}\right) \\
&\times \left[4\left(14Q + \frac{19}{Q}\right) + 3\left(4Q^2 + \frac{1}{Q^2}\right) + 160\right] + 1936 - 4\left(P + \frac{2}{P}\right) \\
&\times \left[12\left(10Q + \frac{21}{Q}\right) + 2\left(20Q^2 + \frac{21}{Q^2}\right) + \left(8Q^3 + \frac{1}{Q^3}\right) + 168\right] = 0.
\end{aligned} \tag{44}
$$

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^2)} \dfrac{\varphi(q^9)}{\varphi(q^{18})}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^2)} \dfrac{\varphi(q^{18})}{\varphi(q^9)}$, *then*

$$\begin{aligned}
&Q^6 + \frac{1}{Q^6} - 136\left(Q^5 + \frac{1}{Q^5}\right) + 2458\left(Q^4 + \frac{1}{Q^4}\right) - 12264\left(Q^3 + \frac{1}{Q^3}\right) \\
&+ 40911\left(Q^2 + \frac{1}{Q^2}\right) - 73104\left(Q + \frac{1}{Q}\right) + 256\left(P^4 + \frac{16}{P^4}\right) + 3840\left(P^3 + \frac{8}{P^3}\right) \\
&+ 17280\left(P^2 + \frac{4}{P^2}\right) + 51072\left(P + \frac{2}{P}\right) + 95532 = 256\left(P^4 + \frac{16}{P^4}\right)\left(Q + \frac{1}{Q}\right) \\
&+ 1152\left(P^3 + \frac{8}{P^3}\right)\left[2\left(Q^2 + \frac{1}{Q^2}\right) - \left(Q + \frac{1}{Q}\right)\right] + 576\left(P^2 + \frac{4}{P^2}\right) \\
&\times \left[25\left(Q + \frac{1}{Q}\right) - 11\left(Q^2 + \frac{1}{Q^2}\right) + 3\left(Q^3 + \frac{1}{Q^3}\right)\right] + 192\left(P + \frac{2}{P}\right) \\
&\times \left[221\left(Q + \frac{1}{Q}\right) - 110\left(Q^2 + \frac{1}{Q^2}\right) + 35\left(Q^3 + \frac{1}{Q^3}\right) - 5\left(Q^4 + \frac{1}{Q^4}\right)\right].
\end{aligned} \tag{45}$$

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^2)}$ *and* $Q = \dfrac{\varphi(q^{10})}{\varphi(q^{20})}$, *then*

$$\begin{aligned}
&Q^{12}(P^{12}+1) + 8P^{11}Q^7(20Q^2 - 16 - 5Q^4) + P^{10}Q^2(8192 - 20480Q^2 \\
&- 6Q^{10} - 6400Q^6 + 17920Q^4 + 800Q^8) + 40P^9Q^5(48Q^2 - 18Q^4 - 32 \\
&+ Q^6) + 5P^8Q^2(20480Q^2 - 8192 + 3Q^{10} - 17920Q^4 + 6428Q^6 - 796Q^8) \\
&+ 16P^7Q^3(100Q^6 - 160 - 700Q^4 + 720Q^2 + 43Q^8) + 4P^6Q^2(-44800Q^2 \\
&+ 17920 - 5Q^{10} - 15820Q^6 + 2300Q^8 + 40384Q^4) + 16P^5Q(480Q^2 - 64 \\
&- 1400Q^4 + 1180Q^6 - 103Q^{10} - 90Q^8) + 5P^4Q^2(25712Q^2 - 25312Q^4 \\
&- 2344Q^8 - 10240 + 3Q^{10} + 12184Q^6) + 40P^3Q(19Q^{10} - 144Q^2 - 72Q^6 \\
&+ 160Q^4 - 28Q^8 + 64) + 2P^2Q^2(2856Q^8 - 15920Q^2 + 18400Q^4 - 11720Q^6 \\
&+ 6400 - 3Q^{10}) + 8PQ(-824Q^6 + 25Q^{10} + 80Q^2 - 160 + 190Q^8 + 688Q^4) \\
&+ 64 + 240Q^4 - 192Q^2 - 12Q^{10} - 160Q^6 + 60Q^8 = 0.
\end{aligned} \tag{46}$$

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^2)} \dfrac{\varphi(q^{11})}{\varphi(q^{22})}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^2)} \dfrac{\varphi(q^{22})}{\varphi(q^{11})}$, *then*

$$\begin{aligned}
&Q^6 - \frac{1}{Q^6} - 286\left(Q^5 + \frac{1}{Q^5}\right) + 11660\left(Q^4 - \frac{1}{Q^4}\right) - 29766\left(Q^3 + \frac{1}{Q^3}\right)\\
&+ 120021\left(Q^2 - \frac{1}{Q^2}\right) - 179388\left(Q + \frac{1}{Q}\right) + 1024\left(P^5 + \frac{32}{P^5}\right)\\
&- 5632\left(P^4 + \frac{16}{P^4}\right)\left(Q + \frac{1}{Q}\right) + 704\left(P^3 + \frac{8}{P^3}\right)\left[45 + 16\left(Q^2 + \frac{1}{Q^2}\right)\right.\\
&\left.-8\left(Q - \frac{1}{Q}\right)\right] + 44\left(P + \frac{2}{P}\right)\left[3386 + 79\left(Q^4 + \frac{1}{Q^4}\right) - 652\left(Q^3 - \frac{1}{Q^3}\right)\right.\\
&\left.+1404\left(Q^2 + \frac{1}{Q^2}\right) - 1548\left(Q - \frac{1}{Q}\right)\right] - 176\left(P^2 + \frac{4}{P^2}\right)\left[56\left(Q^3 + \frac{1}{Q^3}\right)\right.\\
&\left.-129\left(Q^2 - \frac{1}{Q^2}\right) + 370\left(Q + \frac{1}{Q}\right)\right] = 0.
\end{aligned} \tag{47}$$

Modular equations of degree 3

In

- M. S. Mahadeva Naika, S. Chandan Kumar and K. Sushan Bairy, New identities for Ramanujan's cubic continued fraction, Funct. Approx. Comment. Math., 46(1), (2012), 29–44.
- M. S. Mahadeva Naika, S. Chandankumar and K. Sushan Bairy, Some new identities for a continued fraction of order 12, South East Asian J. Math. Math. Sci., (accepted).
- M. S. Mahadeva Naika, S. Chandankumar and K. Sushan Bairy, New identities for ratios of Ramanujan's theta function, (communicated).

We established several new modular equations for the ratios of Ramanujan's theta functions. We also established general formula for the explicit evaluations of ratios of theta functions φ and ψ.

Theorem

If $P := \dfrac{\varphi(q)\varphi(q^2)}{\varphi(q^3)\varphi(q^6)}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^6)}{\varphi(q^2)\varphi(q^3)}$, *then*

$$P + \frac{3}{P} + Q - \frac{1}{Q} = 4. \tag{48}$$

Theorem

If $P := \dfrac{\varphi(q)\varphi(q^4)}{\varphi(q^3)\varphi(q^{12})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{12})}{\varphi(q^4)\varphi(q^3)}$, *then*

$$Q^2 + \frac{1}{Q^2} - 16\left[Q + \frac{1}{Q}\right] + P^2 + \frac{3^2}{P^2} + 2\left[Q - \frac{1}{Q}\right]\left[P + \frac{3}{P}\right] + 20 = 0. \quad (49)$$

Theorem

If $P := \dfrac{\varphi(q)\varphi(q^6)}{\varphi(q^3)\varphi(q^{18})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{18})}{\varphi(q^3)\varphi(q^6)}$, *then*

$$9Q^3 - \frac{1}{Q^3} - 36\left[Q^2 + \frac{1}{Q^2}\right] + 9\left[7Q + \frac{1}{Q}\right] - 48 + P^3 + \frac{3^3}{P^3}$$
$$+ \left[P^2 + \frac{3^2}{P^2}\right]\left\{12 - \left[5Q + \frac{3}{Q}\right]\right\} + \left[P + \frac{3}{P}\right]\left\{3\left[Q^2 + \frac{1}{Q^2}\right]\right. \quad (50)$$
$$\left. -8\left[3Q + \frac{5}{Q}\right] + 51\right\} = 0.$$

Theorem

If $P := \dfrac{\varphi(q)\varphi(q^8)}{\varphi(q^3)\varphi(q^{24})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{24})}{\varphi(q^3)\varphi(q^8)}$, *then*

$$
\begin{aligned}
&Q^4 + \frac{1}{Q^4} + 80\left[Q^3 - \frac{1}{Q^3}\right] + 320\left[Q^2 + \frac{1}{Q^2}\right] - 80\left[Q - \frac{1}{Q}\right] + 884 \\
&+ P^4 + \frac{3^4}{P^4} + 4\left[P^3 + \frac{3^3}{P^3}\right]\left\{4 + \left[Q - \frac{1}{Q}\right]\right\} + 2\left[P^2 + \frac{3^2}{P^2}\right] \\
&\times \left\{16 - 8\left[Q - \frac{1}{Q}\right] + 3\left[Q^2 + \frac{1}{Q^2}\right]\right\} + 4\left[P + \frac{3}{P}\right] \qquad (51) \\
&\times \left\{-52 + 22\left[Q - \frac{1}{Q}\right] - 52\left[Q^2 + \frac{1}{Q^2}\right] + \left[Q^3 - \frac{1}{Q^3}\right]\right\} = 0.
\end{aligned}
$$

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q = \dfrac{\varphi(q^9)}{\varphi(q^{27})}$, *then*

$$\frac{Q^5}{P^5} + 9\left(\frac{Q^3}{P^3} - \frac{Q^4}{P^4}\right) + 9\left(\frac{5Q}{P} + \frac{9P}{Q}\right) + 9\left(Q^4 + \frac{9}{P^4}\right) + 3\left(Q^2 + 3P^2\right)$$

$$\left(PQ + \frac{9}{P^3Q^3}\right) = 135 + 3\left(\frac{Q^2}{P^2} + \frac{9P^2}{Q^2}\right) + \left(P^4Q^4 + \frac{3^4}{P^4Q^4}\right)$$

$$+ 9\left(P^2Q^2 + \frac{3^2}{P^2Q^2}\right). \tag{52}$$

Theorem

If $P := \dfrac{\varphi(q)\varphi(q^{10})}{\varphi(q^3)\varphi(q^{30})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{30})}{\varphi(q^3)\varphi(q^{10})}$, *then*

$$\begin{aligned}
&Q^6 + \frac{1}{Q^6} + 160\left[Q^5 + \frac{1}{Q^5}\right] + 2468\left[Q^4 + \frac{1}{Q^4}\right] + 160\left[Q^3 + \frac{1}{Q^3}\right] + 14864 \\
&+ 1745\left[Q^2 + \frac{1}{Q^2}\right] + 8320\left[Q + \frac{1}{Q}\right] + P^6 + \frac{3^6}{P^6} + \left[P + \frac{3}{P}\right]\left(6\left[Q^5 - \frac{1}{Q^5}\right]\right. \\
&\left.+640\left\{\left[Q^2 - \frac{1}{Q^2}\right] - \left[Q^4 - \frac{1}{Q^4}\right]\right\} + 470\left[Q^3 - \frac{1}{Q^3}\right] - 580\left[Q - \frac{1}{Q}\right]\right) \\
&+ \left[P^2 + \frac{3^2}{P^2}\right]\left(15\left[Q^4 + \frac{1}{Q^4}\right] - 576\left[Q^3 + \frac{1}{Q^3}\right] - 1960\left[Q^2 + \frac{1}{Q^2}\right]\right. \\
&\left.-1600\left[Q + \frac{1}{Q}\right] - 1415\right) + \left[P^3 + \frac{3^3}{P^3}\right]\left(20\left[Q^3 - \frac{1}{Q^3}\right] + 384\left[Q^2 - \frac{1}{Q^2}\right]\right. \\
&\left.+350\left[Q - \frac{1}{Q}\right]\right) + \left[P^4 + \frac{3^4}{P^4}\right]\left(15\left[Q^2 + \frac{1}{Q^2}\right] + 160\left[Q + \frac{1}{Q}\right] + 324\right) \\
&+ 6\left[P^5 + \frac{3^5}{P^5}\right]\left(Q - \frac{1}{Q}\right) = 0.
\end{aligned} \tag{53}$$

Theorem

If $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q := \dfrac{\varphi(q^{5/2})}{\varphi(q^{15/2})}$, *then*

$$\begin{aligned}
&\frac{P^6}{Q^6}+\frac{Q^6}{P^6}-160\left[\frac{P^5}{Q^5}+\frac{Q^5}{P^5}\right]+2468\left[\frac{P^4}{Q^4}+\frac{Q^4}{P^4}\right]-160\left[\frac{P^3}{Q^3}+\frac{Q^3}{P^3}\right]\\
&+1745\left[\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\right]-8320\left[\frac{P}{Q}+\frac{Q}{P}\right]+\left[P^6Q^6+\frac{3^6}{P^6Q^6}\right]+14864\\
&+\left[PQ+\frac{3}{PQ}\right]\left[580\left[\frac{P}{Q}-\frac{Q}{P}\right]+640\left[\frac{P^2}{Q^2}-\frac{Q^2}{P^2}\right]-470\left[\frac{P^3}{Q^3}-\frac{Q^3}{P^3}\right]\right.\\
&\left.-640\left[\frac{P^4}{Q^4}-\frac{Q^4}{P^4}\right]-6\left[\frac{P^5}{Q^5}-\frac{Q^5}{P^5}\right]\right]+\left[P^2Q^2+\frac{3^2}{P^2Q^2}\right]\left[1600\left[\frac{P}{Q}+\frac{Q}{P}\right]\right.\\
&\left.-1415-1960\left[\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\right]+576\left[\frac{P^3}{Q^3}+\frac{Q^3}{P^3}\right]+15\left[\frac{P^4}{Q^4}+Q^4P^4\right]\right]\\
&+\left[P^3Q^3+\frac{3^3}{P^3Q^3}\right]\left[-350\left[\frac{P}{Q}-\frac{Q}{P}\right]+384\left[\frac{P^2}{Q^2}-\frac{Q}{P^2}\right]-20\left[\frac{P^3}{Q^3}-\frac{Q^3}{P^3}\right]\right]\\
&+\left[P^4Q^4+\frac{3^4}{P^4Q^4}\right]\left[+324-160\left[\frac{P}{Q}+\frac{Q}{P}\right]+15\left[\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\right]\right]\\
&-6\left[P^5Q^5+\frac{3^5}{P^5Q^5}\right]\left[\frac{P}{Q}-\frac{Q}{P}\right]=0.
\end{aligned} \tag{54}$$

THEOREM

If $P := \dfrac{\varphi(q)\varphi(q^{12})}{\varphi(q^3)\varphi(q^{36})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{36})}{\varphi(q^3)\varphi(q^{12})}$, *then*

$$
\begin{aligned}
&81Q^6 + \frac{1}{Q^6} - \frac{288}{Q^5} + 252\left[3Q^4 + \frac{35}{Q^4}\right] + 96\left[9Q^3 + \frac{68}{Q^3}\right] + 52560 \\
&+ 9\left[1993Q^2 - \frac{551}{Q^2}\right] - 1152\left[6Q - \frac{49}{Q}\right] + P^6 + \frac{3^6}{P^6} - \left[P^5 + \frac{3^5}{P^5}\right] \\
&\times\left\{48 + 2\left[5Q + \frac{3}{Q}\right]\right\} + \left[P^4 + \frac{3^4}{P^4}\right]\left\{852 + 96\left[2Q + \frac{3}{Q}\right] + \left[31Q^2 + \frac{15}{Q^2}\right]\right\} \\
&+ \left[P^3 + \frac{3^3}{P^3}\right]\left\{102\left[13Q - \frac{5}{Q}\right] + 96\left[Q^2 + \frac{5}{Q^2}\right] - 4\left[3Q^3 + \frac{5}{Q^3}\right] - 2320\right\} \\
&+ \left[P^2 + \frac{3^2}{P^2}\right]\left\{4761 - 192\left[32Q + \frac{5}{Q}\right] + 8\left[63Q^2 - \frac{881}{Q^2}\right] - 192\left[3Q^3 + \frac{10}{Q^3}\right]\right. \\
&\left.-3\left[27Q^4 - \frac{5}{Q^4}\right]\right\} + \left[P + \frac{3}{P}\right]\left\{36\left[231Q - \frac{431}{Q}\right] - 96\left[57Q^2 - \frac{131}{Q^2}\right]\right. \\
&\left.+30\left[21Q^3 + \frac{67}{Q^3}\right] - 16\left[27Q^4 + \frac{163}{Q^4}\right] + 6\left[9Q^5 - \frac{1}{Q^5}\right] - 21696\right\} = 0.
\end{aligned}
\tag{55}
$$

Theorem

If $P = \dfrac{\psi(q)\psi(q^{17})}{q^{9/2}\psi(q^3)\psi(q^{51})}$ *and* $Q = \dfrac{\psi(q)\psi(q^{51})}{q^{-4}\psi(q^3)\psi(q^{17})}$, *then*

$$\begin{aligned}
&Q^9 - \frac{1}{Q^9} + 34\left(Q^8 + \frac{1}{Q^8}\right) + 272\left(Q^7 - \frac{1}{Q^7}\right) + 238\left(Q^6 + \frac{1}{Q^6}\right) \\
&- 595\left(Q^5 - \frac{1}{Q^5}\right) - 510\left(Q^4 + \frac{1}{Q^4}\right) + 16303\left(Q^3 - \frac{1}{Q^3}\right) \\
&- 5202\left(Q^2 + \frac{1}{Q^2}\right) - 26911\left(Q - \frac{1}{Q}\right) + \left(P^8 + \frac{3^8}{P^8}\right) + 20230 = \\
&17\left\{\left(P^2 + \frac{3^2}{P^2}\right)\left[7\left(Q^6 + \frac{1}{Q^6}\right) + 28\left(Q^5 - \frac{1}{Q^5}\right) + 34\left(Q^4 + \frac{1}{Q^4}\right)\right.\right. \\
&\left. -168\left(Q^3 - \frac{1}{Q^3}\right) + 160\left(Q^2 + \frac{1}{Q^2}\right) + 378\left(Q - \frac{1}{Q}\right) - 210\right] \\
&- \left(P^4 + \frac{3^4}{P^4}\right)\left[5\left(Q^4 + \frac{1}{Q^4}\right) + 21\left(Q^3 - \frac{1}{Q^3}\right) - 35\left(Q^2 + \frac{1}{Q^2}\right)\right. \\
&\left.\left. -14\left(Q - \frac{1}{Q}\right) - 30\right] - \left(P^6 + \frac{3^6}{P^6}\right)\left[\left(Q^2 + \frac{1}{Q^2}\right) - 2\left(Q - \frac{1}{Q}\right) - 2\right]\right\}.
\end{aligned} \tag{56}$$

Theorem If $P = \dfrac{\psi(q)\psi(q^{19})}{q^5\psi(q^3)\psi(q^{57})}$ and $Q = \dfrac{\psi(q)\psi(q^{57})}{q^{-9/2}\psi(q^3)\psi(q^{19})}$, then

$$\begin{aligned}
&Q^{10} - \frac{1}{Q^{10}} + \left(P^9 + \frac{3^9}{P^9}\right) + 19\left\{26\left(Q^8 - \frac{1}{Q^8}\right) + 135\left(Q^6 - \frac{1}{Q^6}\right)\right. \\
&+ 1916\left(Q^4 - \frac{1}{Q^4}\right) + 1144\left(Q^2 - \frac{1}{Q^2}\right) - \left(P^7 + \frac{3^7}{P^7}\right)\left[\left(Q^2 - \frac{1}{Q^2}\right) - 4\right] \\
&- 5\left(P^6 + \frac{3^6}{P^6}\right)\left[\left(Q^2 - \frac{1}{Q^2}\right)\right] + 3\left(P^5 + \frac{3^5}{P^5}\right)\left[2\left(Q^4 + \frac{1}{Q^4}\right)\right. \\
&\left.-14\left(Q^2 + \frac{1}{Q^2}\right) + 11\right] + 2\left(P^4 + \frac{3^4}{P^4}\right)\left[26\left(Q^4 - \frac{1}{Q^4}\right) - \left(Q^2 - \frac{1}{Q^2}\right)\right] \\
&- 5\left(P^2 + \frac{3^2}{P^2}\right)\left[13\left(Q^6 - \frac{1}{Q^6}\right) - 4\left(Q^4 - \frac{1}{Q^4}\right) + 104\left(Q^2 - \frac{1}{Q^2}\right)\right] \\
&- \left(P^3 + \frac{3^3}{P^3}\right)\left[12\left(Q^6 + \frac{1}{Q^6}\right) - 65\left(Q^4 + \frac{1}{Q^4}\right) - 591\right. \\
&\left.+200\left(Q^2 + \frac{1}{Q^2}\right)\right] + 3\left(P + \frac{3}{P}\right)\left[2\left(Q^8 + \frac{1}{Q^8}\right) - 29\left(Q^6 + \frac{1}{Q^6}\right)\right. \\
&\left.\left.+7\left(Q^4 + \frac{1}{Q^4}\right) - 504\left(Q^2 + \frac{1}{Q^2}\right) + 697\right]\right\} = 0.
\end{aligned} \tag{57}$$

Modular equations of degree 4

In

- M. S. Mahadeva Naika, K. Sushan Bairy and M. Manjunatha, Some new modular equations of degree four and their explicit evaluations, Eur. J. Pure Appl. Math., 3 (6), 924-947 (2010).

the authors have established several new modular equations of degree 4 and established general formula for explicit evaluation of $h_{4,n}$.

THEOREM

If $X = \dfrac{\varphi(q)\varphi(q^2)}{\varphi(q^4)\varphi(q^8)}$ *and* $Y = \dfrac{\varphi(q)\varphi(q^8)}{\varphi(q^4)\varphi(q^2)}$, *then*

$$X^2 - 2X^3Y + X^4Y^2 - 4XY + 4Y^2 - 4XY^3 + 4X^2Y^2 - 2X^3Y^3 + 2X^2Y^4 = 0. \tag{58}$$

Theorem

If $X = \dfrac{\varphi(q)\varphi(q^3)}{\varphi(q^4)\varphi(q^{12})}$ *and* $Y = \dfrac{\varphi(q)\varphi(q^{12})}{\varphi(q^4)\varphi(q^3)}$, *then*

$$Y^2 + \frac{1}{Y^2} - 12\left(Y + \frac{1}{Y}\right) + 12\left(\sqrt{X} + \frac{2}{\sqrt{X}}\right)\left(\sqrt{Y} + \frac{1}{\sqrt{Y}}\right) = 4\left(X + \frac{4}{X}\right) + 30. \qquad (59)$$

Theorem

If $X = \dfrac{\varphi(q)\varphi(q^4)}{\varphi(q^4)\varphi(q^{16})}$ *and* $Y = \dfrac{\varphi(q)\varphi(q^{16})}{\varphi(q^4)\varphi(q^4)}$, *then*

$$X^4Y^2 + 6X^3Y + X^2 + 16Y^2 + 24XY - 4\sqrt{XY}\left(X^3Y + X^2 + 2X + 8Y\right) = 0. \qquad (60)$$

Theorem

If $X = \dfrac{\varphi(q)\varphi(q^5)}{\varphi(q^4)\varphi(q^{20})}$ *and* $Y = \dfrac{\varphi(q)\varphi(q^{20})}{\varphi(q^4)\varphi(q^5)}$, *then*

$$Y^3 + \frac{1}{Y^3} - 70\left[Y^2 + \frac{1}{Y^2}\right] - 785\left[Y + \frac{1}{Y}\right] + 160\left[\sqrt{Y^3} + \frac{1}{\sqrt{Y^3}}\right] \times \left[\sqrt{X} + \frac{2}{\sqrt{X}}\right] + 80\left[\sqrt{Y} + \frac{1}{\sqrt{Y}}\right]\left[\sqrt{X^3} + \frac{8}{\sqrt{X^3}} + 10\left(\sqrt{X} + \frac{2}{\sqrt{X}}\right)\right] \quad (61)$$
$$= 80\left[X + \frac{4}{X}\right]\left[5 + 2\left(Y + \frac{1}{Y}\right)\right] + 16\left[X^2 + \frac{16}{X^2}\right] + 1620.$$

Theorem

If $X = \dfrac{\varphi(q)\varphi(q^7)}{\varphi(q^4)\varphi(q^{28})}$ *and* $Y = \dfrac{\varphi(q)\varphi(q^{28})}{\varphi(q^4)\varphi(q^7)}$, *then*

$$
\begin{aligned}
&Y^4 + \frac{1}{Y^4} - 280\left(Y^3 + \frac{1}{Y^3}\right) - 28\left(Y^2 + \frac{1}{Y^2}\right)\left[349 + 78\left(X + \frac{4}{X}\right)\right] \\
&- 56\left(Y + \frac{1}{Y}\right)\left[1079 + 310\left(X + \frac{4}{X}\right) + 24\left(X^2 + \frac{16}{X^2}\right)\right] \\
&+ 112\left(\sqrt{Y} + \frac{1}{\sqrt{Y}}\right)\left[515\left(\sqrt{X} + \frac{2}{\sqrt{X}}\right) + 90\left(\sqrt{X^3} + \frac{8}{\sqrt{X^3}}\right)\right. \\
&\left.+4\left(\sqrt{X^5} + \frac{32}{\sqrt{X^5}}\right)\right] + 56\left(\sqrt{Y^3} + \frac{1}{\sqrt{Y^3}}\right)\left[40\left(\sqrt{X^3} + \frac{8}{\sqrt{X^3}}\right)\right. \\
&\left.+313\left(\sqrt{X} + \frac{2}{\sqrt{X}}\right)\right] + 1176\left(\sqrt{Y^5} + \frac{1}{\sqrt{Y^5}}\right)\left(\sqrt{X} + \frac{2}{\sqrt{X}}\right) \\
&= 16\left[4\left(X^3 + \frac{64}{X^3}\right) + 203\left(X^2 + \frac{16}{X^2}\right) + 2023\left(X + \frac{4}{X}\right)\right] + 106330.
\end{aligned}
\tag{62}
$$

THEOREM

If $X = \dfrac{\varphi(q)\varphi(q^9)}{\varphi(q^4)\varphi(q^{36})}$ *and* $Y = \dfrac{\varphi(q)\varphi(q^{36})}{\varphi(q^4)\varphi(q^9)}$, *then*

$$\begin{aligned}
&Y^6 + \frac{1}{Y^6} - 908\left[Y^5 + \frac{1}{Y^5}\right] - 83582\left[Y^4 + \frac{1}{Y^4}\right] - 1369692\left[Y^3 + \frac{1}{Y^3}\right] \\
&- 3\left[Y^2 + \frac{1}{Y^2}\right]\left\{2657883 + 96832\left[X^2 + \frac{16}{X^2}\right]\right\} - 24\left[Y + \frac{1}{Y}\right] \\
&\times\left\{892353 + 289628\left[X + \frac{4}{X}\right]\right\} + 17323008\left[\sqrt{Y} + \frac{1}{\sqrt{Y}}\right]\left[\sqrt{X} + \frac{2}{\sqrt{X}}\right] \\
&+ 1831776\left[\sqrt{Y^3} + \frac{1}{\sqrt{Y^3}}\right]\left[\sqrt{X^3} + \frac{8}{\sqrt{X^3}}\right] + 21504\left[\sqrt{Y^5} + \frac{1}{\sqrt{Y^5}}\right] \\
&\times\left[\sqrt{X^5} + \frac{32}{\sqrt{X^5}}\right] - 29469924 = 128\left\{2\left[X^4 + \frac{256}{X^4}\right]\right. \\
&\left.+420\left[X^3 + \frac{64}{X^3}\right] + 9987\left[X^2 + \frac{16}{X^2}\right] + 75426\left[X + \frac{4}{X}\right]\right\}.
\end{aligned} \tag{63}$$

Modular equations of degree 5

In

- M. S. Mahadeva Naika, B. N. Dharmendra and S. Chandankumar, New modular relations for Ramanujan's parameter $\mu(q)$, Int. J. Pure Appl. Math., 74 (4) (2012), 413–435.
- M. S. Mahadeva Naika, B. N. Dharmendra and S. Chandankumar, On some new modular relations for Ramanujan's $\kappa(q)$ function and $\nu(q)$ function, (communicated).

We established several new $P-Q$ modular equations for the ratios of Ramanujan's theta function. Using these equations, we established some general formulas for explicit evaluations for ratios of Ramanujan's theta functions. We also established several new modular relations connecting $\kappa(q)$ with $\kappa(q^n)$, $\nu(q)$ with $\nu(q^n)$, $\mu(q)$ with $\mu(q^n)$ and new modular relations connecting $\kappa(q)$, $\nu(q)$ and $\mu(q)$, where μ, κ and ν are parameters involving Rogers-Ramanujan continued fraction.

Theorem

If $P := \dfrac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)}$ *and* $Q := \dfrac{\varphi(q)}{\varphi(q^5)}$, *then*

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 4 = P^2 + \frac{5}{P^2}. \tag{64}$$

Theorem

If $P := \dfrac{\psi(-q)\psi(-q^2)}{q^{\frac{3}{2}}\psi(-q^5)\psi(-q^{10})}$ *and* $Q := \dfrac{\psi(-q)\psi(-q^{10})}{q^{-\frac{1}{2}}\psi(-q^5)\psi(-q^2)}$, *then*

$$Q^4 + \frac{1}{Q^4} + \left(Q^2 + \frac{1}{Q^2}\right) = P^2 + \frac{25}{P^2} + \left(P + \frac{5}{P}\right)\left[5\left(Q + \frac{1}{Q}\right) - \left(Q^3 + \frac{1}{Q^3}\right)\right] + 30. \quad (65)$$

Theorem

If $P := \dfrac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)}$ *and* $Q := \dfrac{\psi(q^4)}{q^2\psi(q^{20})}$, *then*

$$\frac{P^4}{Q^4} + \frac{Q^4}{P^4} + 24\left(\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right) + 8\left(P^2Q^2 + \frac{25}{P^2Q^2}\right) - 20\left(Q^2 + \frac{5}{Q^2}\right) + 120$$
$$+ 3\left(P^4 + \frac{25}{P^4}\right) - 32\left(P^2 + \frac{5}{P^2}\right) = P^4\left(Q^2 + \frac{3}{Q^2}\right) + \frac{5}{P^4}\left(3Q^2 + \frac{25}{Q^2}\right). \tag{66}$$

Theorem

If $P := \dfrac{\psi(-q)}{q^{\frac{1}{2}}\psi(-q^5)}$ *and* $Q := \dfrac{\psi(-q^4)}{q^2\psi(-q^{20})}$, *then*

$$\begin{aligned}
&Q^8 + \frac{1}{Q^8} - 19\left(Q^6 + \frac{1}{Q^6}\right) - 419\left(Q^4 + \frac{1}{Q^4}\right) - 1327\left(Q^2 + \frac{1}{Q^2}\right) - 2332 \\
&+ \left(P + \frac{5}{P}\right)\left[\left(Q^7 + \frac{1}{Q^7}\right) - 44\left(Q^5 + \frac{1}{Q^5}\right) - 295\left(Q^3 + \frac{1}{Q^3}\right) - 672\left(Q + \frac{1}{Q}\right)\right. \\
&- \left(P^2 + \frac{5^2}{P^2}\right)\left[28\left(Q^4 + \frac{1}{Q^4}\right) + 109\left(Q^2 + \frac{1}{Q^2}\right) + 132\right] - \left(P^3 + \frac{5^3}{P^3}\right) \\
&\times \left[13\left(Q + \frac{1}{Q}\right) + 9\left(Q^3 + \frac{1}{Q^3}\right)\right] - \left(P^4 + \frac{5^4}{P^4}\right)\left[\left(Q^2 + \frac{1}{Q^2}\right)\right] = 0.
\end{aligned} \tag{67}$$

Theorem

If $P := \dfrac{\psi(-q)}{q^{1/2}\psi(-q^5)}$ *and* $Q := \dfrac{\psi(-q^5)}{q^{5/2}\psi(-q^{25})}$ *then*

$$\frac{Q^3}{P^3} - \frac{5Q^2}{P^2} - \frac{15Q}{P} - 5\left(PQ + \frac{5}{PQ}\right) - 5\left(Q^2 + \frac{5}{P^2}\right) - P^2Q^2 + \frac{5^2}{P^2Q^2} - 15 = 0. \tag{68}$$

Theorem

If $P := \dfrac{\psi(-q)\psi(-q^7)}{q^4\psi(-q^5)\psi(-q^{35})}$ *and* $Q := \dfrac{\psi(-q)\psi(-q^{35})}{q^{-3}\psi(-q^5)\psi(-q^7)}$, *then*

$$Q^4 - \frac{1}{Q^4} + 14\left[\left(Q^3 + \frac{1}{Q^3}\right) + \left(Q^2 - \frac{1}{Q^2}\right) + 10\left(Q + \frac{1}{Q}\right)\right] + P^3 + \frac{5^3}{P^3}$$
$$+ 7\left\{\left(P^2 + \frac{5^2}{P^2}\right)\left(Q + \frac{1}{Q}\right) + \left(P + \frac{5}{P}\right)\left[2\left(Q^2 + \frac{1}{Q^2}\right) + 9\right]\right\} = 0. \quad (69)$$

Theorem

If $P := \dfrac{\psi(-q)\psi(-q^{11})}{q^6\psi(-q^5)\psi(-q^{55})}$ *and* $Q := \dfrac{\psi(-q)\psi(-q^{55})}{q^{-5}\psi(-q^5)\psi(-q^{11})}$, *then*

$$\begin{aligned}
&Q^6+\frac{1}{Q^6}-33\left(Q^5+\frac{1}{Q^5}\right)-99\left(Q^4+\frac{1}{Q^4}\right)-1529\left(Q^3+\frac{1}{Q^3}\right)\\
&-1683\left(Q^2+\frac{1}{Q^2}\right)-8800\left(Q+\frac{1}{Q}\right)-6534-\left(P^5+\frac{5^5}{P^5}\right)\\
&-11\left\{\left(P^4+\frac{5^4}{P^4}\right)\left(Q+\frac{1}{Q}\right)+\left(P^3+\frac{5^3}{P^3}\right)\left[11+4\left(Q^2+\frac{1}{Q^2}\right)\right]\right.\\
&+\left(P^2+\frac{5^2}{P^2}\right)\left[18+56\left(Q+\frac{1}{Q}\right)+3\left(Q^2+\frac{1}{Q^2}\right)+8\left(Q^3+\frac{1}{Q^3}\right)\right]\\
&+\left(P+\frac{5}{P}\right)\left[324+126\left(Q+\frac{1}{Q}\right)+160\left(Q^2+\frac{1}{Q^2}\right)+18\left(Q^3+\frac{1}{Q^3}\right)\right.\\
&\left.\left.+9\left(Q^4+\frac{1}{Q^4}\right)\right]+\left(P^3+\frac{5^3}{P^3}\right)\left[11+4\left(Q^2+\frac{1}{Q^2}\right)\right]\right\}=0.
\end{aligned} \tag{70}$$

Theorem

If $P := \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q := \dfrac{\varphi(q^2)}{\varphi(q^{10})}$, *then*

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + (PQ)^2 + \left(\frac{5}{PQ}\right)^2 + 16\left(\frac{P}{Q} - \frac{Q}{P}\right)$$
$$= 2\left(P^2 + \frac{5}{P^2}\right) + 2\left(Q^2 + \frac{5}{Q^2}\right) + 4. \quad (71)$$

Theorem

If $P := \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q := \dfrac{\varphi(-q^4)}{\varphi(-q^{20})}$, *then*

$$\frac{P^4}{Q^4}+\frac{Q^4}{P^4}+24\left(\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\right)+8\left(P^2Q^2+\frac{5^2}{P^2Q^2}\right)+3\left(Q^4+\frac{5^2}{Q^4}\right)+120$$
$$=20\left(P^2+\frac{5}{P^2}\right)+32\left(Q^2+\frac{5}{Q^2}\right)+\left(P^2Q^4+\frac{125}{P^2Q^4}\right)+3\left(\frac{5P^2}{Q^4}+\frac{Q^4}{P^2}\right). \tag{72}$$

Theorem

If $P := \dfrac{\varphi(q)\varphi(q^4)}{\varphi(q^5)\varphi(q^{20})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{20})}{\varphi(q^5)\varphi(q^4)}$, *then*

$$Q^4 + \frac{1}{Q^4} - 112\left(Q^3 + \frac{1}{Q^3}\right) + 1440\left(Q^2 + \frac{1}{Q^2}\right) - 3184\left(Q + \frac{1}{Q}\right) + 7316$$
$$= 8\left(P + \frac{1}{P}\right)\left[22\left(Q^2 + \frac{1}{Q^2}\right) - 31\left(Q + \frac{1}{Q}\right) + 170\right] - 2\left(P^2 + \frac{5^2}{P^2}\right)$$
$$\times\left[3\left(Q^2 + \frac{1}{Q^2}\right) + 24\left(Q + \frac{1}{Q}\right) + 64\right] + 4\left(P^3 + \frac{5^3}{P^3}\right)\left[\left(Q + \frac{1}{Q}\right) + 4\right]. \tag{73}$$

Theorem

If $P := \frac{\varphi(q)}{\varphi(q^5)}$ *and* $Q := \frac{\varphi(q^5)}{\varphi(q^{25})}$, *then*

$$\begin{aligned} &\frac{Q^3}{P^3} - \frac{5Q^2}{P^2} - \frac{15Q}{P} + 5\left(PQ + \frac{5}{PQ}\right) + 5\left(Q^2 + \frac{5}{P^2}\right) \\ &= P^2Q^2 + \frac{5^2}{P^2Q^2} + 15. \end{aligned} \tag{74}$$

Theorem

If $P := \dfrac{\phi(q)\phi(q^7)}{\phi(q^5)\phi(q^{35})}$ *and* $Q := \dfrac{\phi(q)\phi(q^{35})}{\phi(q^5)\phi(q^7)}$, *then*

$$Q^4 - \frac{1}{Q^4} - 14\left[\left(Q^3 + \frac{1}{Q^3}\right) - \left(Q^2 - \frac{1}{Q^2}\right) + 10\left(Q + \frac{1}{Q}\right)\right] + P^3 + \frac{5^3}{P^3}$$
$$= 7\left\{\left(P^2 + \frac{5^2}{P^2}\right)\left(Q + \frac{1}{Q}\right) - \left(P + \frac{5}{P}\right)\left[2\left(Q^2 + \frac{1}{Q^2}\right) + 9\right]\right\}. \tag{75}$$

Theorem

If $P = \dfrac{\phi(q)\phi(q^{11})}{\phi(q^5)\phi(q^{55})}$ *and* $Q = \dfrac{\phi(q)\phi(q^{55})}{\phi(q^5)\phi(q^{11})}$, *then*

$$
\begin{aligned}
&Q^6 + \frac{1}{Q^6} + 33\left(Q^5 + \frac{1}{Q^5}\right) - 99\left(Q^4 + \frac{1}{Q^4}\right) + 1529\left(Q^3 + \frac{1}{Q^3}\right) \\
&- 1683\left(Q^2 + \frac{1}{Q^2}\right) + 8800\left(Q + \frac{1}{Q}\right) = 6534 + \left(P^5 + \frac{5^5}{P^5}\right) \\
&- 11\left\{\left(P^4 + \frac{5^4}{P^4}\right)\left(Q + \frac{1}{Q}\right) - \left(P^3 + \frac{5^3}{P^3}\right)\left[11 + 4\left(Q^2 + \frac{1}{Q^2}\right)\right]\right. \\
&- \left(P^2 + \frac{5^2}{P^2}\right)\left[18 - 56\left(Q + \frac{1}{Q}\right) + 3\left(Q^2 + \frac{1}{Q^2}\right) - 8\left(Q^3 + \frac{1}{Q^3}\right)\right] \\
&- \left(P + \frac{5}{P}\right)\left[324 - 126\left(Q + \frac{1}{Q}\right) + 160\left(Q^2 + \frac{1}{Q^2}\right) - 18\left(Q^3 + \frac{1}{Q^3}\right)\right. \\
&\left.\left. +9\left(Q^4 + \frac{1}{Q^4}\right)\right] - \left(P^3 + \frac{5^3}{P^3}\right)\left[11 + 4\left(Q^2 + \frac{1}{Q^2}\right)\right]\right\}.
\end{aligned}
\tag{76}
$$

Modular equations of degree 9

In

- M. S. Mahadeva Naika, K. Sushan Bairy and S. Chandan Kumar, On some explicit evaluation of the ratios of Ramanujan's theta-function, (communicated).
- M. S. Mahadeva Naika, S. Chandan Kumar and K. Sushan Bairy, Modular equations for the ratios of Ramanujan's theta function ψ and evaluations, New Zealand J. Math., 40, (2010), 33–48.
- M. S. Mahadeva Naika, S. Chandan Kumar and M. Manjunatha, On Some new modular equations of degree 9 and its applications, Adv. Stud. Contemp. Math., (to appear).

we establish several new modular equations of degree 9 using Ramanujan's modular equations. We also establish several general formulas for explicit evaluations of $h_{9,n},\ h'_{9,n},\ l_{9,n}$ and $l'_{9,n}$. As an application, we establish some explicit evaluations for Ramanujan's cubic continued fraction.

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^9)}$ *and* $Q = \dfrac{\varphi(-q)}{\varphi(-q^9)}$, *then*

$$\frac{P}{Q} + \frac{Q}{P} + PQ + \frac{9}{PQ} + 4 = 2\left(\sqrt{PQ} + \frac{3}{\sqrt{PQ}}\right)\left(\sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}}\right). \quad (77)$$

Proof. From Entry 56, of Ch. 25, p. 210 (Ramanujan's Notebook Part IV, by Berndt), we have

$$M^3 + N^3 = M^2N^2 + 3MN, \quad (78)$$

where $M = \dfrac{f(-q)}{q^{1/3}f(-q^9)}$ and $N = \dfrac{f(-q^2)}{q^{2/3}f(-q^{18})}$.
Replacing q by $-q$ in equation (78), we deduce that

$$Y^3 - X^3 = X^2Y^2 - 3XY, \quad (79)$$

where

$$X = \frac{f(q)}{q^{1/3}f(q^9)} \quad \text{and} \quad Y = \frac{f(-q^2)}{q^{2/3}f(-q^{18})}. \quad (80)$$

From the Theorem 2.2 of [M. S. Mahadeva Naika, Some theorems on Ramanujan's cubic continued fraction and related identities. Tamsui Oxf. J. Math. Sci. 24(3) (2008), 243–256], we have

$$\frac{f^3(-q)}{qf^3(-q^9)} = \frac{\varphi^2(-q)}{\varphi^2(-q^9)}\frac{\varphi(-q)-3\varphi(-q^9)}{\varphi(-q)-\varphi(-q^9)}, \tag{81}$$

$$\frac{f^3(-q^2)}{q^2f^3(-q^{18})} = \frac{\varphi(-q)}{\varphi(-q^9)}\left(\frac{\varphi(-q)-3\varphi(-q^9)}{\varphi(-q)-\varphi(-q^9)}\right)^2. \tag{82}$$

Using the equation (81) by changing q to $-q$ along with the equation (82) in the above equation (79), we deduce that

$$X^3 = P^2\left(\frac{3-P}{P-1}\right) \quad \text{and} \quad Y^3 = Q\left(\frac{Q-3}{Q-1}\right)^2. \tag{83}$$

Equation (79) can be rewritten as

$$a^2 - 3a - A = 0, \tag{84}$$

where $a = XY$ and $A = Y^3 - X^3$.
Solving the equation (84) for a and cubing both sides, we find that

$$8X^3Y^3 = (3+m)^3, \quad \text{where } m = \pm\sqrt{9+4A}. \tag{85}$$

Eliminating m from the above equation (85), we deduce that

$$(-Q+P)(P^2Q^2-2PQ^2+Q^2-6Q+9+4PQ-6P-2P^2Q+P^2)$$
$$(PQ^2-Q^2+3Q-4PQ+3P+P^2Q-P^2)(P^4Q^3-3P^4Q^2+3P^4Q$$
$$-P^4-4P^3Q^3+12P^3Q^2-8P^3Q+6P^2Q^3-Q^4-24P^2Q^2+18P^2Q$$
$$-8PQ^3+36PQ^2-36PQ+9Q^3-27Q^2+27Q)=0. \tag{86}$$

By examining the behavior of the above factors near $q=0$, we can find a neighborhood about the origin, where the second factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof.

Theorem

If $P = \dfrac{\psi(q)}{q\psi(q^9)}$ and $Q = \dfrac{\psi(-q)}{q\psi(-q^9)}$, then

$$\frac{P}{Q} + \frac{Q}{P} + \frac{3}{P} + P = \frac{3}{Q} + Q + 4. \tag{87}$$

Theorem

If $P = \dfrac{\psi(-q)}{q\psi(-q^9)}$ and $Q = \dfrac{\varphi(q)}{\varphi(q^9)}$, then

$$Q + PQ = 3 + P. \tag{88}$$

Theorem

If $P = \dfrac{\psi(q)}{q\psi(q^9)}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^9)}$, *then*

$$\frac{P}{Q} + \frac{Q}{P} + 2 = \frac{3}{P} + P. \qquad (89)$$

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^9)}$ *and* $Q = \dfrac{\varphi(-q^2)}{\varphi(-q^{18})}$, *then*

$$\frac{P}{Q} + \frac{Q}{P} + 2 = \frac{3}{Q} + Q. \qquad (90)$$

THEOREM

If $P = \dfrac{\psi(q)}{q\psi(q^9)}$ *and* $Q = \dfrac{\psi(q^2)}{q^2\psi(q^{18})}$, *then*

$$\frac{P}{Q} + \frac{Q}{P} + 2 = \frac{3}{P} + P. \tag{91}$$

THEOREM

If $P = \dfrac{\varphi(q)\varphi(q^2)}{\varphi(q^9)\varphi(q^{18})}$ *and* $Q = \dfrac{\varphi(q)\varphi(q^{18})}{\varphi(q^9)\varphi(q^2)}$, *then*

$$Q^2 + \frac{1}{Q^2} + \left(P^2 + \frac{9^2}{P^2}\right) + 2\left(P + \frac{9}{P}\right)\left[4 + 3\left(Q + \frac{1}{Q}\right)\right] = 8\left(Q + \frac{1}{Q}\right)$$
$$+ 4\left[\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) - \left(\sqrt{P} + \frac{3}{\sqrt{P}}\right)\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 3\right]. \tag{92}$$

Theorem

If $P = \dfrac{\psi(-q)}{q\psi(-q^9)}$ *and* $Q = \dfrac{\psi(-q^2)}{q^2\psi(-q^{18})}$, *then*

$$\frac{P^2}{Q^2} + \frac{Q^2}{P^2} + \frac{P}{Q} + \frac{Q}{P} = PQ + \frac{9}{PQ} + \left(P + \frac{3}{P}\right)\left(3 - \frac{P}{Q}\right) + \left(Q + \frac{3}{Q}\right)\left(3 - \frac{Q}{P}\right) + 12. \tag{93}$$

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^9)}$ *and* $Q = \dfrac{\varphi(q^3)}{\varphi(q^{27})}$, *then*

$$\left(3 - P - \frac{3}{P}\right)\left(3 - Q - \frac{3}{Q}\right) = \left(\frac{Q}{P}\right)^2. \tag{94}$$

Theorem

If $P = \dfrac{\varphi(q)\varphi(q^5)}{\varphi(q^9)\varphi(q^{45})}$ *and* $Q = \dfrac{\varphi(q)\varphi(q^{45})}{\varphi(q^9)\varphi(q^5)}$, *then*

$$Q^3 + \frac{1}{Q^3} - 15\left(Q^2 + \frac{1}{Q^2}\right) - 45\left(Q + \frac{1}{Q}\right) - \left(P^2 + \frac{81}{P^2}\right) - 10\left(P + \frac{9}{P}\right)$$
$$\times \left[2 + Q + \frac{1}{Q}\right] + 5\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) + 15\left(\sqrt{P} + \frac{3}{\sqrt{P}}\right)$$
$$\times \left[\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 2\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right)\right] = 40. \tag{95}$$

Theorem

If $P = \dfrac{\varphi(q)}{\varphi(q^9)}\dfrac{\varphi(q^7)}{\varphi(q^{63})}$ *and* $Q = \dfrac{\varphi(q)\varphi(q^{63})}{\varphi(q^9)\varphi(q^7)}$, *then*

$$
\begin{aligned}
&Q^4 + \frac{1}{Q^4} - 35\left(Q^3 + \frac{1}{Q^3}\right) - 413\left(Q^2 + \frac{1}{Q^2}\right) - 1379\left(Q + \frac{1}{Q}\right) - 1694 \\
&- \left(P^3 + \frac{9^3}{P^3}\right) - 7\left(P^2 + \frac{9^2}{P^2}\right)\left[7 + 3\left(Q + \frac{1}{Q}\right)\right] - 21\left(P + \frac{9}{P}\right) \\
&\left[21 + 14\left(Q + \frac{1}{Q}\right) + 3\left(Q^2 + \frac{1}{Q^2}\right)\right] + 7\left(\sqrt{P^5} + \frac{3^5}{\sqrt{P^5}}\right)\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) \\
&+ 63\left[\sqrt{P} + \frac{3}{\sqrt{P}}\right]\left[7\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 14\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) + \left(\sqrt{Q^5} + \frac{1}{\sqrt{Q^5}}\right)\right] \\
&+ 21\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left[2\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 7\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right)\right] = 0.
\end{aligned}
\tag{96}
$$

Theorem

If $P = \dfrac{f(-q)f(-q^5)}{q^2 f(-q^9)f(-q^{45})}$ *and* $Q = \dfrac{f(-q)f(-q^{45})}{q^{-4/3} f(-q^9)f(-q^5)}$ *then*

$$Q^3 + \frac{1}{Q^3} = 10 + \left(P^2 + \frac{9^2}{P^2}\right) + 5\left(\sqrt{P} + \frac{3}{\sqrt{P}}\right)\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 5\left(P + \frac{9}{P}\right). \tag{97}$$

Theorem

If $P = \dfrac{f(-q)f(-q^7)}{q^{8/3} f(-q^9)f(-q^{63})}$ *and* $Q = \dfrac{f(-q)f(-q^{63})}{q^{-2} f(-q^9)f(-q^7)}$ *then*

$$Q^4 + \frac{1}{Q^4} - 14\left(Q^3 + \frac{1}{Q^3}\right) + 28\left(Q^2 + \frac{1}{Q^2}\right) + 7\left(Q + \frac{1}{Q}\right) = P^3 + \frac{9^3}{P^3} + 7\left(\sqrt{P^3} + \frac{27}{\sqrt{P^3}}\right)\left[\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) - \left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right)\right] + 98. \tag{98}$$

Theorem

If $P = \dfrac{f(-q)f(-q^{11})}{q^4 f(-q^9)f(-q^{99})}$ and $Q = \dfrac{f(-q)f(-q^{99})}{q^{-10/3} f(-q^9)f(-q^{11})}$ then

$$Q^6 + \frac{1}{Q^6} = 165\left(P + \frac{9}{P}\right) + 66\left(P^2 + \frac{9^2}{P^2}\right) + 11\left(P^3 + \frac{9^3}{P^3}\right) + 1848$$
$$+ \left(P^5 + \frac{9^5}{P^5}\right) + 22\left(Q^3 + \frac{1}{Q^3}\right)\left[2\left(P^2 + \frac{9^2}{P^2}\right) + 3\left(P + \frac{9}{P}\right) + 26\right]$$
$$+ 11\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right)\left[15\left(\sqrt{P} + \frac{3}{\sqrt{P}}\right) + 14\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\right.$$
$$\left. + \left(\sqrt{P^5} + \frac{3^5}{\sqrt{P^5}}\right) + \left(\sqrt{P^7} + \frac{3^7}{\sqrt{P^7}}\right)\right] + 55\left(\sqrt{P} + \frac{3}{P}\right)\left(\sqrt{Q^9} + \frac{1}{\sqrt{Q^9}}\right). \tag{99}$$

Theorem

If $P = \dfrac{f(-q)f(-q^{13})}{q^{14/3}f(-q^9)f(-q^{117})}$ *and* $Q = \dfrac{f(-q)f(-q^{117})}{q^{-4}f(-q^9)f(-q^{13})}$ *then*

$$Q^7 + \frac{1}{Q^7} - 65\left(Q^6 + \frac{1}{Q^6}\right) + 910\left(Q^5 + \frac{1}{Q^5}\right) - 1417\left(Q^4 + \frac{1}{Q^4}\right)$$
$$- 6994\left(Q^3 + \frac{1}{Q^3}\right) + 10049\left(Q^2 + \frac{1}{Q^2}\right) + 6981\left(Q + \frac{1}{Q}\right) = 17472$$
$$P^6 + \frac{9^6}{P^6} - 13\left(\sqrt{P^9} + \frac{3^9}{\sqrt{P^9}}\right)\left[\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) - \left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right)\right]$$
$$- 13\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left[139\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) - 179\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right)\right.$$
$$\left. -2\left(\sqrt{Q^5} + \frac{1}{\sqrt{Q^5}}\right) + 52\left(\sqrt{Q^7} + \frac{1}{\sqrt{Q^7}}\right) + 10\left(\sqrt{Q^9} + \frac{1}{\sqrt{Q^9}}\right)\right]$$
$$+ 13\left(P^3 + \frac{9^3}{P^3}\right)\left[5\left(Q^3 + \frac{1}{Q^3}\right) - 12\left(Q^2 + \frac{1}{Q^2}\right) - 6\left(Q + \frac{1}{Q}\right) + 26\right]. \tag{100}$$

Applications of Modular equations

$P - Q$ identities are extremely useful for explicit evaluations of Rogers - Ramanujan, Ramanujan - Selberg, Ramanujan's cubic continued fractions and Ramanujan - Weber class invariants.

In the following theorem, we restate Theorems 1 and 3 which gives us formulas connecting g_n and g_{kn}, for $k = 9, 25, 49, 121$.

(i)

$$2\sqrt{2}\left[g_n^3 g_{9n}^3 + \frac{1}{g_n^3 g_{9n}^3}\right] = \frac{g_{9n}^6}{g_n^6} - \frac{g_n^6}{g_{9n}^6}, \tag{101}$$

(ii)

$$2\left[g_n^2 g_{25n}^2 + \frac{1}{g_n^2 g_{25n}^2}\right] = \frac{g_{25n}^3}{g_n^3} - \frac{g_n^3}{g_{25n}^3}, \tag{102}$$

(iii)

$$16\sqrt{2}\left[g_n^9 g_{49n}^9 + \frac{1}{g_n^9 g_{49n}^9}\right] + 168\left[g_n^6 g_{49n}^6 + \frac{1}{g_n^6 g_{49n}^6}\right]$$
$$+ 336\sqrt{2}\left[g_n^3 g_{49n}^3 + \frac{1}{g_n^3 g_{49n}^3}\right] + 658 = \frac{g_{49n}^{12}}{g_n^{12}} + \frac{g_n^{12}}{g_{49n}^{12}}, \tag{103}$$

(iv)

$$32\left[g_n^{10}g_{121n}^{10}+\frac{1}{g_n^{10}g_{121n}^{10}}\right]+352\left[g_n^{8}g_{121n}^{8}+\frac{1}{g_n^{8}g_{121n}^{8}}\right]$$
$$+1672\left[g_n^{6}g_{121n}^{6}+\frac{1}{g_n^{6}g_{121n}^{6}}\right]+4576\left[g_n^{4}g_{121n}^{4}+\frac{1}{g_n^{4}g_{121n}^{4}}\right]$$
$$+8096\left[g_n^{2}g_{121n}^{2}+\frac{1}{g_n^{2}g_{121n}^{2}}\right]+9744=\frac{g_{121n}^{12}}{g_n^{12}}+\frac{g_n^{12}}{g_{121n}^{12}}, \tag{104}$$

(v)

$$2\left[g_ng_{4n}g_{9n}g_{36n}+\frac{1}{g_ng_{4n}g_{9n}g_{36n}}\right]=\frac{g_{9n}^2g_{36n}^2}{g_n^2g_{4n}^2}+\frac{g_n^2g_{4n}^2}{g_{9n}^2g_{36n}^2}, \tag{105}$$

(vi)

$$8\left[g_n^3g_{4n}^3g_{49n}^3g_{196n}^3+\frac{1}{g_n^3g_{4n}^3g_{49n}^3g_{196n}^3}\right]$$
$$+28\left[g_n^2g_{4n}^2g_{49n}^2g_{196n}^2+\frac{1}{g_n^2g_{4n}^2g_{49n}^2g_{196n}^2}\right]$$
$$+56\left[g_ng_{4n}g_{49n}g_{196n}+\frac{1}{g_ng_{4n}g_{49n}g_{196n}}\right]+70=\frac{g_{49n}^4g_{196n}^4}{g_n^4g_{4n}^4}+\frac{g_n^4g_{4n}^4}{g_{49n}^4g_{196n}^4}. \tag{106}$$

(vii)

$$g_{4n}g_{36n}=\sqrt{g_n^4g_{9n}^4+g_ng_{9n}\sqrt{g_n^6g_{9n}^6+2}}, \tag{107}$$

(viii)

$$g_{4n}g_{100n} = 2^{-1/4}\left[g_n^2 g_{25n}^2\left\{g_n^2 g_{25n}^2\left(4g_n^4 g_{25n}^4 + 4^{1/3}\right)\right.\right.$$
$$\left.\left. + \sqrt{g_n^4 g_{25n}^4\left(4g_n^4 g_{25n}^4 + 4^{1/3}\right)^2 + 16}\right\}\right]^{1/4}, \qquad (108)$$

(ix)

$$g_{4n}g_{196n} = \frac{g_n^2 g_{49n}^2 + \sqrt{g_n g_{49n}\left(2g_n^3 g_{49n}^3 + 2^{3/2}\right)}}{\sqrt{2}}, \qquad (109)$$

(x)

$$g_{4n}g_{484n} =$$

$$\sqrt{g_n g_{121n}\left\{g_n g_{121n}\left(4+g_n^2 g_{121n}^2\right)+\sqrt{g_n^2 g_{121n}^2\left(4+g_n^2 g_{121n}^2\right)^2+2}\right\}} \tag{110}$$

and

(xi)

$$g_{4n}g_{2116n} = \frac{1}{\sqrt{2}}$$

$$\times\left[g_n g_{529n}\left(g_n g_{529n}+\sqrt{2}\right)+\sqrt{g_n^2 g_{529n}^2\left(g_n g_{529n}+\sqrt{2}\right)^2+2\sqrt{2}g_n g_{529n}}\right]. \tag{111}$$

In the following theorem, we restate Theorems 1 and 2 which gives us formulas connecting G_n with G_{kn}, for $k = 9, 25, 49, 121$.

(i)

$$2\sqrt{2}\left[G_n^3 G_{9n}^3 - \frac{1}{G_n^3 G_{9n}^3}\right] = \frac{G_{9n}^6}{G_n^6} + \frac{G_n^6}{G_{9n}^6}, \tag{112}$$

(ii)

$$2\left[G_n^2 G_{25n}^2 - \frac{1}{G_n^2 G_{25n}^2}\right] = \frac{G_{25n}^3}{G_n^3} + \frac{G_n^3}{G_{25n}^3}, \tag{113}$$

(iii)

$$16\sqrt{2}\left[G_n^9 G_{49n}^9 + \frac{1}{G_n^9 G_{49n}^9}\right] - 168\left[G_n^6 G_{49n}^6 + \frac{1}{G_n^6 G_{49n}^6}\right] + 336\sqrt{2}\left[G_n^3 G_{49n}^3 + \frac{1}{G_n^3 G_{49n}^3}\right] - 658 = \frac{G_{49n}^{12}}{G_n^{12}} + \frac{G_n^{12}}{G_{49n}^{12}}, \tag{114}$$

(iv)

$$32\left[G_n^{10} G_{121n}^{10} + \frac{1}{G_n^{10} G_{121n}^{10}}\right] - 352\left[G_n^8 G_{121n}^8 + \frac{1}{G_n^8 G_{121n}^8}\right]$$

$$+1672\left[G_n^6G_{121n}^6+\frac{1}{G_n^6G_{121n}^6}\right]-4576\left[G_n^4G_{121n}^4+\frac{1}{G_n^4G_{121n}^4}\right]$$

$$+8096\left[G_n^2G_{121n}^2+\frac{1}{G_n^2G_{121n}^2}\right]-9744=\frac{G_{121n}^{12}}{G_n^{12}}+\frac{G_n^{12}}{G_{121n}^{12}}, \tag{115}$$

(v)

$$2\left[G_ng_{4n}G_{9n}g_{36n}+\frac{1}{G_ng_{4n}G_{9n}g_{36n}}\right]=\frac{G_{9n}^2g_{36n}^2}{G_n^2g_{4n}^2}+\frac{G_n^2g_{4n}^2}{G_{9n}^2g_{36n}^2}, \tag{116}$$

and (vi)

$$8\left[G_n^3g_{4n}^3G_{49n}^3g_{196n}^3+\frac{1}{G_n^3g_{4n}^3G_{49n}^3g_{196n}^3}\right]$$

$$-28\left[G_n^2g_{4n}^2G_{49n}^2g_{196n}^2+\frac{1}{G_n^2g_{4n}^2G_{49n}^2g_{196n}^2}\right]$$

$$+56\left[G_ng_{4n}G_{49n}g_{196n}+\frac{1}{G_ng_{4n}G_{49n}g_{196n}}\right]-70$$

$$= \frac{G_{49n}^4 g_{196n}^4}{G_n^4 g_{4n}^4} + \frac{G_n^4 g_{4n}^4}{G_{49n}^4 g_{196n}^4}. \tag{117}$$

The author has also established the explicit evaluations of g_n for $n = \ 2/9, \ 2/3, \ 2/5, \ 2/25, \ 2/7, \ 2/11$.

In

- Jinhee Yi, Theta-function identities and the explicit formulas for theta-function and their applications, J. Math. Anal. Appl., 292(2004), 381-400.

the author introduced two parameterization $h_{k,n}$ and $h'_{k,n}$ for Ramanujan theta function φ as follows

$$h_{k,n} = \frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})}, \quad h'_{k,n} = \frac{\varphi(-e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(-e^{-\pi\sqrt{nk}})} \tag{118}$$

and established some properties and several explicit evaluations of $h_{k,n}$ for different positive rational values of n and k. The author evaluates $h_{2,2}$, $h_{2,4}$, $h_{2,8}$, $h_{2,1/2}$, $h_{2,1/4}$, $h_{2,1/8}$.

In

- J. Yi, Yang Lee and Dae Hyun Paek, The explicit formulas and evaluations of Ramanujan's theta-function ψ, J. Math. Anal. Appl., 321(2006), 157–181.

and

- N. D. Baruah and Nipen Saikia, Two parameters for Ramanujan's theta–functions and their explicit values, Rocky Mountain J. Math., 37 (6) (2007), 1747–1790.

the authors have defined two parameters $l_{k,n}$ and $l'_{k,n}$ as follows:

$$l_{k,n} = \frac{\psi(-e^{-\pi\sqrt{n/k}})}{k^{1/4}q^{(k-1)/8}\psi(-e^{-\pi\sqrt{nk}})} \quad and \quad l'_{k,n} = \frac{\psi(e^{-\pi\sqrt{n/k}})}{k^{1/4}q^{(k-1)/8}\psi(e^{-\pi\sqrt{nk}})}.$$

They have established several properties as well as explicit evaluations of $l_{k,n}$ and $l'_{k,n}$ for different positive rational values of n and k.

Using the modular equations of degree 2, we established several explicit evaluations of $h_{2,n}$.

$$h_{2,4} = 1+\sqrt{2}-\sqrt{1+\sqrt{2}},$$

$$h_{2,1/4} = \frac{1+\sqrt{\sqrt{2}-1}}{\sqrt{2}},$$

$$h_{2,16} = 5+3\sqrt{2}+(\sqrt{2}+1)^{5/2}-\sqrt{76+54\sqrt{2}+(488\sqrt{2}+690)(\sqrt{2}-1)^{5/2}},$$

$$h_{2,1/16} = \left(\frac{\sqrt{\sqrt{2}+1}}{\sqrt{2}}\right)\left(\sqrt{\sqrt{2}+1}+\sqrt{2\left(\sqrt{\sqrt{2}+1}-\sqrt{2}\right)}-1\right),$$

$$h_{2,5} = \left(\sqrt{2}-1\right)\left(2+\sqrt{5}\right)^{1/2},$$

$$h_{2,1/5} = \left(\sqrt{2}+1\right)\left(-2+\sqrt{5}\right)^{1/2},$$

$$h_{2,20} = \frac{(2+\sqrt{5})^{1/2}(\sqrt{2}+1)^2\left[\sqrt{2}-1-\sqrt{-3+\sqrt{10}}\right]}{(-3+\sqrt{10})},$$

$$h_{2,1/20} = \frac{(-2+\sqrt{5})^{1/2}(\sqrt{2}-1)^2(-3+\sqrt{10})}{\left[\sqrt{2}-1-\sqrt{-3+\sqrt{10}}\right]},$$

$$h_{2,7} = \sqrt{p_1 - \sqrt{q_1}},$$

$$h_{2,1/7} = \sqrt{p_1 + \sqrt{q_1}},$$

where

$$p_1 = -\sqrt{2}\left(5+4\sqrt{2}-\sqrt{65+46\sqrt{2}}\right)$$

and

$$q_1 = \left(\left(13 - 5\sqrt{2}\right) - 4\sqrt{7\left(-11 + 8\sqrt{2}\right)}\right)\left(\sqrt{2} + 1\right)^5.$$

$$h_{2,8} = (\sqrt{2} + 1)^2 \sqrt{\sqrt{2} - 2\sqrt{2\sqrt{2}}(\sqrt{2} - 1)},$$

$$h_{2,1/8} = \sqrt{\frac{1 + 2\sqrt[4]{2}(\sqrt{2} - 1)}{2}},$$

$$h_{2,9} = (\sqrt{3} + \sqrt{2})(2 - \sqrt{3}),$$
$$h_{2,1/9} = (\sqrt{3} - \sqrt{2})(\sqrt{3} + 2),$$
$$h_{2,36} = (\sqrt{3} + \sqrt{2})(1 + \sqrt{2})^3(2 + \sqrt{3})^2 \left[2 - \sqrt{3} - \sqrt{5\sqrt{2} - 7}\right],$$
$$h_{2,1/36} = \frac{(\sqrt{3} - \sqrt{2})(\sqrt{2} - 1)^3(2 - \sqrt{3})^2}{\left[2 - \sqrt{3} - \sqrt{5\sqrt{2} - 7}\right]},$$

$$h_{2,6} = \sqrt{8\sqrt{2} - 6\sqrt{3} + 4\sqrt{6} - 10},$$

$$h_{2,1/6} = \frac{\sqrt{1 + 2\sqrt{2} + \sqrt{3}}}{2},$$

$$h_{2,2/3} = \frac{(\sqrt{1 + 2\sqrt{2} + \sqrt{3}} - \sqrt{-1 + 2\sqrt{2} - \sqrt{3}})\sqrt{2}}{\sqrt{3} + 1},$$

$$h_{2,3/2} = \frac{(\sqrt{1 + 2\sqrt{2} + \sqrt{3}} + \sqrt{-1 + 2\sqrt{2} - \sqrt{3}})}{2\sqrt{2}},$$

$$h_{2,24} = \frac{\sqrt{-10 + 8\sqrt{2} - 6\sqrt{3} + 4\sqrt{6}} - \sqrt{10 - 7\sqrt{2} + 6\sqrt{3} - 4\sqrt{6}}}{(\sqrt{2} - 1)^2(\sqrt{3} - \sqrt{2})^2},$$

$$h_{2,1/24} = \frac{\sqrt{-10 + 8\sqrt{2} - 6\sqrt{3} + 4\sqrt{6}} + \sqrt{10 - 7\sqrt{2} + 6\sqrt{3} - 4\sqrt{6}}}{\sqrt{2}}.$$

We have established explicit evaluations of Ramanujan–Göllnitz–Gordon continued fraction $H(q)$, where

$$H(q) := \frac{q^{1/2}}{1+q} \, {}_{+} \, \frac{q^2}{1+q^3} \, {}_{+} \, \frac{q^4}{1+q^5} \, {}_{+} \, \frac{q^6}{1+q^7} \, {}_{+} \, \cdots, \qquad |\, q \,| < 1, \quad (119)$$

using the values of $h_{2,n}$.

We have established the following relation connecting $h_{2,n}$ and Ramanujan–Selberg continued fraction .

LEMMA

For any positive rational number n, we have

$$V^4(e^{-\pi\sqrt{2n}}) = \frac{\sqrt{2}h_{2,n}^2 - 1}{4}, \tag{120}$$

where

$$\begin{aligned} V(q) &:= \frac{q^{1/8}}{1+} \frac{q}{1+} \frac{q^2+q}{1+} \frac{q^3}{1+} \frac{q^4+q^2}{1+} \cdots \\ &= \frac{q^{1/8}(-q^2;q^2)_\infty}{(-q;q^2)_\infty}, \qquad |q|<1, \end{aligned} \tag{121}$$

We have established the following relation connecting $h_{2,n}$ and a continued fraction of Eisenstein .

LEMMA

For any positive rational number n, we have

$$E^4(e^{-\pi\sqrt{n/2}}) = \frac{\sqrt{2} - h_{2,n}^2}{h_{2,n}^2}, \tag{122}$$

where

$$E(q) := \frac{(q;q^2)_\infty}{(-q;q^2)_\infty} = \frac{1}{1} + \frac{2q}{1-q^2} + \frac{-q^3-q}{1+q^4} + \frac{q^5+q^3}{1-q^6} + \cdots, \quad \text{for } |q| < 1. \tag{123}$$

Using the modular equations of degree 3, we established several explicit evaluations of $h_{3,n}$ and $l_{3,n}$.

$$h_{3,9} = \frac{\sqrt[3]{4} - \sqrt[3]{2} + 1}{\sqrt{3}},$$
$$h_{3,1/9} = \frac{1 + \sqrt[3]{2}}{\sqrt{3}},$$
$$l_{3,9} = \frac{(\sqrt[3]{2} + 1)^2}{\sqrt{3}},$$
$$l_{3,1/9} = \frac{(\sqrt[3]{4} - 1)}{\sqrt{3}},$$

$$h_{3,17}^2 = \frac{2 + \sqrt{17} - (142 + 34\sqrt{17})^{1/3} + (2 + 2\sqrt{17})^{1/3}}{3},$$
$$h_{3,1/17}^2 = \frac{\sqrt{17} - 2 + (142 - 34\sqrt{17})^{1/3} + (-2 + 2\sqrt{17})^{1/3}}{3},$$
$$l_{3,17}^2 = 4 + \sqrt{17} + 2(29 + 7\sqrt{17})^{1/3} + 2(37 + 9\sqrt{17})^{1/3},$$
$$l_{3,1/17}^2 = -4 + \sqrt{17} - 2(29 - 7\sqrt{17})^{1/3} + 2(-37 + 9\sqrt{17})^{1/3},$$

$$h_{3,19}^4 = (26 - 15\sqrt{3})(2\sqrt{19} + 5\sqrt{3}),$$
$$h_{3,1/19}^4 = (26 + 15\sqrt{3})(2\sqrt{19} - 5\sqrt{3}),$$
$$l_{3,19}^4 = (26 + 15\sqrt{3})(2\sqrt{19} + 5\sqrt{3}),$$
$$l_{3,1/19}^4 = (26 - 15\sqrt{3})(2\sqrt{19} - 5\sqrt{3}),$$

Using the modular equations of degree 4, we established several explicit evaluations of $h_{4,n}$.

$$
\begin{aligned}
h_{4,4} &= \left(2-\sqrt{2\sqrt{2}}\right)(\sqrt{2}+1), \\
h_{4,1/4} &= \frac{(\sqrt{2}+\sqrt[4]{2})}{2}, \\
h_{4,2} &= 1+\sqrt{2}-\sqrt{\sqrt{2}+1}, \\
h_{4,1/2} &= \frac{1+\sqrt{\sqrt{2}-1}}{\sqrt{2}},
\end{aligned}
$$

$$
\begin{aligned}
h_{4,9} &= 5-3\sqrt{2}+3\sqrt{3}-2\sqrt{6}-\sqrt{93-66\sqrt{2}+54\sqrt{3}-38\sqrt{6}}, \\
h_{4,1/9} &= 5-3\sqrt{2}+3\sqrt{3}-2\sqrt{6}+\sqrt{93-66\sqrt{2}+54\sqrt{3}-38\sqrt{6}}, \\
h_{4,3} &= \left(\sqrt{2}-1\right)\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right), \\
h_{4,1/3} &= \left(\sqrt{2}+1\right)\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right),
\end{aligned}
$$

$$h_{4,25} = (\sqrt{2}-1)^4 \left(18+8\sqrt{2}+3\sqrt{5}-2\sqrt{27\sqrt{5}+12\sqrt{10}-30\sqrt{2}-20}\right),$$

$$h_{4,1/25} = (\sqrt{2}-1)^4 \left(18+8\sqrt{2}+3\sqrt{5}+2\sqrt{27\sqrt{5}+12\sqrt{10}-30\sqrt{2}-20}\right),$$

$$h_{4,5} = \sqrt{\frac{17+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}} - \sqrt{\frac{15+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}},$$

$$h_{4,1/5} = \sqrt{\frac{17+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}} + \sqrt{\frac{15+7\sqrt{5}-4\sqrt{29+13\sqrt{5}}}{2}},$$

$$h_{4,7} = \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)\left(2\sqrt{2}-\sqrt{7}\right),$$

$$h_{4,1/7} = \left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)\left(2\sqrt{2}+\sqrt{7}\right),$$

$$h_{4,8} = \sqrt{2}(\sqrt{2}+1)^{3/2}\left[\sqrt{2}+\sqrt{\sqrt{2}+1}-\sqrt{4+\sqrt{2+10\sqrt{2}}}\right]$$

$$h_{4,1/8} = \frac{(\sqrt{2}-1)^{3/2}}{\sqrt{2}\left[\sqrt{2}+\sqrt{\sqrt{2}+1}-\sqrt{4+\sqrt{2+10\sqrt{2}}}\right]}$$

$$h_{4,9} = 5-3\sqrt{2}+3\sqrt{3}-2\sqrt{6}-\sqrt{93-66\sqrt{2}+54\sqrt{3}-38\sqrt{6}},$$

$$h_{4,1/9} = 5-3\sqrt{2}+3\sqrt{3}-2\sqrt{6}+\sqrt{93-66\sqrt{2}+54\sqrt{3}-38\sqrt{6}},$$

$$h_{4,13} = -\sqrt{\frac{1279+355\sqrt{13}+12\sqrt{a_1}}{2}}+\sqrt{\frac{1281+355\sqrt{13}+12\sqrt{a_1}}{2}},$$

$$h_{4,1/13} = \sqrt{\frac{1279+355\sqrt{13}+12\sqrt{a_1}}{2}}+\sqrt{\frac{1281+355\sqrt{13}+12\sqrt{a_1}}{2}},$$

where $a_1 = 22733 + 6305\sqrt{13}$,

$$
\begin{aligned}
h_{4,17} &= \frac{v - \sqrt{v^2 - 4}}{2}, \\
h_{4,1/17} &= \frac{v + \sqrt{v^2 - 4}}{2},
\end{aligned}
$$

where $v =$
$$\left(\sqrt{2} - 1\right)^2 \left[11 + \sqrt{2} - \sqrt{17}\left(\sqrt{2} - 1\right) + 2\sqrt{\sqrt{17}\left(9 - 2\sqrt{2}\right) - 2\left(13\sqrt{2} - 6\right)}\right].$$

Using the modular equations of degree 9, we established several explicit evaluations of $h_{9,n}$, $h'_{9,n}$, $l_{9,n}$ and $l'_{9,n}$.

$$h_{9,2} = (\sqrt{3}+\sqrt{2})(2-\sqrt{3}),$$
$$h_{9,1/2} = (\sqrt{3}-\sqrt{2})(\sqrt{3}+2),$$
$$h_{9,4} = 5-3\sqrt{2}+3\sqrt{3}-2\sqrt{6}-\sqrt{93-66\sqrt{2}+54\sqrt{3}-38\sqrt{6}},$$
$$h_{9,1/4} = 5-3\sqrt{2}+3\sqrt{3}-2\sqrt{6}+\sqrt{93-66\sqrt{2}+54\sqrt{3}-38\sqrt{6}},$$

$$h_{9,3} = \frac{\sqrt[3]{4}-\sqrt[3]{2}+1}{\sqrt{3}},$$
$$h_{9,1/3} = \frac{1+\sqrt[3]{2}}{\sqrt{3}},$$
$$h_{9,9} = 2[1+(2-\sqrt{3})^{2/3}]-(-5+3\sqrt{3})(2+\sqrt{3})^{2/3}-\sqrt{3},$$

$$h_{9,5} = \frac{(\sqrt{5}-\sqrt{3})(\sqrt{3}-1)}{2},$$

$$h_{9,1/5} = \frac{(\sqrt{5}+\sqrt{3})(\sqrt{3}+1)}{2},$$

$$h_{9,25} = \frac{(7-3\sqrt{5})(4+\sqrt{15})}{2} - \frac{a}{2},$$

$$h_{9,1/25} = \frac{(7-3\sqrt{5})(4+\sqrt{15})}{2} + \frac{a}{2},$$

$$h_{9,7} = \frac{\sqrt{10+2\sqrt{21}}}{4} - \frac{\sqrt{2\sqrt{21}-6}}{4},$$

$$h_{9,1/7} = \frac{\sqrt{10+2\sqrt{21}}}{4} + \frac{\sqrt{2\sqrt{21}-6}}{4},$$

$$h_{9,11} = \sqrt{a+b-3-\sqrt{(a+b)^2-6(a+b)-8}},$$

$$h_{9,1/11} = \sqrt{a+b-3+\sqrt{(a+b)^2-6(a+b)-8}},$$

where $a=\sqrt[3]{19+3\sqrt{33}}$ and $b=\sqrt[3]{19-3\sqrt{33}}$.

$$h_{9,13}=\sqrt{\frac{77-21\sqrt{13}+44\sqrt{3}-12\sqrt{39}}{2}}-\sqrt{\frac{75-21\sqrt{13}+44\sqrt{3}-12\sqrt{39}}{2}},$$

$$h_{9,1/13}=\sqrt{\frac{77-21\sqrt{13}+44\sqrt{3}-12\sqrt{39}}{2}}+\sqrt{\frac{75-21\sqrt{13}+44\sqrt{3}-12\sqrt{39}}{2}}.$$

$$h'_{9,1} = \sqrt{3} + 1 - \sqrt{3 + 2\sqrt{3}},$$

$$h'_{9,2} = \sqrt{3} - \sqrt{2},$$

$$h'_{9,1/2} = (\sqrt{3} - \sqrt{2})(2 - \sqrt{3}),$$

$$h'_{9,3} = \frac{x^2 + 2^{5/3}x + 2^{1/3}(5 - 3\sqrt{3})}{\sqrt{3}}, \quad \text{where} \quad x = 2 - \sqrt{3},$$

$$h'_{9,1/3} = \frac{2^{1/3}(1 - \sqrt{3})}{\sqrt{3}},$$

$$h'_{9,5} = \frac{15 + 10\sqrt{3} + 7\sqrt{5} + 4\sqrt{15}}{2} - \frac{a}{\sqrt{2}},$$

where $a = \sqrt{3(163 + 94\sqrt{3} + 73\sqrt{5} + 42\sqrt{15})}$.

$$h'_{9,4} = \frac{\sqrt{3}+1-\sqrt{2}\sqrt[4]{3}}{2},$$

$$h'_{9,8} = \frac{\sqrt{3}-1}{\sqrt{2}},$$

$$h'_{9,12} = \frac{2^{2/3}(1-\sqrt{3})-2\sqrt{3}+2^{4/3}(1+2^{2/3})}{2\sqrt{3}},$$

$$h'_{9,20} = \frac{\sqrt{(2+\sqrt{3})(7+3\sqrt{5})}}{4} - \frac{\sqrt{(\sqrt{5}+1)(6+3\sqrt{3})}}{2},$$

$$h'_{9,28} = \frac{2\sqrt{7}+4\sqrt{3}-3-\sqrt{21}}{4} + \frac{\sqrt{(-3+\sqrt{21})(5+\sqrt{21}-2\sqrt{7}-4\sqrt{3})^2}}{4\sqrt{2}},$$

$$l_{9,2} = \frac{\sqrt{3}+1}{\sqrt{2}},$$

$$l_{9,1/2} = \frac{\sqrt{3}-1}{\sqrt{2}},$$

$$l_{9,4} = \frac{3+\sqrt{3}+\sqrt{2}+\sqrt{2(3+\sqrt{3})(\sqrt{3}+\sqrt{2})}}{2\sqrt{2}},$$

$$l_{9,1/4} = \frac{3+\sqrt{3}+\sqrt{2}-\sqrt{2(3+\sqrt{3})(\sqrt{3}+\sqrt{2})}}{2\sqrt{2}},$$

$$l_{9,8} = \frac{\sqrt{13+5\sqrt{6}+7\sqrt{3}+9\sqrt{2}}}{\sqrt{2}} + \frac{\sqrt{15+6\sqrt{6}+9\sqrt{3}+12\sqrt{2}}}{\sqrt{2}},$$

$$l_{9,1/8} = \frac{\sqrt{13+5\sqrt{6}+7\sqrt{3}+9\sqrt{2}}}{\sqrt{2}} - \frac{\sqrt{15+6\sqrt{6}+9\sqrt{3}+12\sqrt{2}}}{\sqrt{2}},$$

$$l_{9,3} = \frac{(\sqrt[3]{2}+1)^2}{\sqrt{3}},$$

$$l_{9,5} = \frac{(\sqrt{5}+\sqrt{3})(\sqrt{3}+1)}{2},$$

$$l_{9,1/5} = \frac{(\sqrt{5}-\sqrt{3})(\sqrt{3}-1)}{2},$$

$$l_{9,7} = \sqrt{10+2\sqrt{21}} + \sqrt{9+2\sqrt{21}},$$

$$l_{9,1/7} = \sqrt{10+2\sqrt{21}} - \sqrt{9+2\sqrt{21}},$$

$$l_{9,9} = x + 2x^{2/3} + (\sqrt{3}-1)x^{4/3}, \text{ where } x = 2+\sqrt{3},$$

$$l_{9,5} = \frac{(\sqrt{3}+1)(\sqrt{5}+\sqrt{3})}{2},$$

$$l_{9,1/5} = \frac{(\sqrt{3}-1)(\sqrt{5}-\sqrt{3})}{2},$$

$$l_{9,25} = \frac{(7+3\sqrt{5})(4+\sqrt{15})}{2} + \frac{a}{2},$$

$$l_{9,1/25} = \frac{(7+3\sqrt{5})(4+\sqrt{15})}{2} - \frac{a}{2},$$

where $a = \sqrt{2910 + 752\sqrt{15} + 1302\sqrt{5} + 1680\sqrt{3}}.$

$$l'_{9,3} = \frac{(\sqrt{3}+1)^2 + (\sqrt{3}+1)\sqrt[3]{4} + \sqrt[3]{16}}{2\sqrt{3}},$$

$$l'_{9,7} = \frac{3 + 2\sqrt{7} + 4\sqrt{3} + \sqrt{21} + \sqrt{9+2\sqrt{21}}(2+3\sqrt{3}-\sqrt{7})}{4},$$

$$l'_{9,1} = \frac{\sqrt{3}+1+\sqrt{2}\sqrt[4]{3}}{2},$$

$$l'_{1/9,1} = \frac{\sqrt{3}+1-\sqrt{2}\sqrt[4]{3}}{2},$$

$$l'_{9,2} = \sqrt{3}+\sqrt{2},$$

$$l'_{9,1/2} = \frac{\sqrt{3}+1}{\sqrt{2}},$$

$$l'_{1/9,1/2} = \sqrt{3}-\sqrt{2},$$

$$l'_{1/9,2} = \frac{\sqrt{3}-1}{\sqrt{2}},$$

$$l'_{9,5} = \frac{(3+\sqrt{5})(1+\sqrt{3})}{4} + \sqrt{\frac{(3\sqrt{3}+6)(\sqrt{5}+1)}{4}},$$

$$l'_{1/9,1/5} = \frac{(3+\sqrt{5})(1+\sqrt{3})}{4} - \sqrt{\frac{(3\sqrt{3}+6)(\sqrt{5}+1)}{4}},$$

$$l'_{1/9,5} = \frac{(3+\sqrt{5})(-1+\sqrt{3})}{4} - \sqrt{\frac{(-3\sqrt{3}+6)(\sqrt{5}+1)}{4}},$$

$$l'_{9,1/5} = \frac{(3+\sqrt{5})(-1+\sqrt{3})}{4} + \sqrt{\frac{(-3\sqrt{3}+6)(\sqrt{5}+1)}{4}},$$

$$l'_{9,4} = 1+\sqrt{3}+\sqrt{2\sqrt{3}+3},$$
$$l'_{1/9,4} = 1+\sqrt{3}-\sqrt{2\sqrt{3}+3},$$
$$l'_{9,8} = (2+\sqrt{3})(\sqrt{3}+\sqrt{2}),$$
$$l'_{9,12} = (2+\sqrt{3})^2+\sqrt[3]{2}(5+3\sqrt{3})+[\sqrt[3]{2}(\sqrt{3}+1)]^2,$$
$$l'_{9,28} = 7\sqrt{7}+11\sqrt{3}+4\sqrt{21}+18+(2+\sqrt{7})(2+\sqrt{3})\sqrt{9+2\sqrt{21}}.$$

We established the explicit evaluation of Ramanujan's cubic continued fraction using the values of $h_{9,n}$ and $h'_{9,n}$.

$$C(q) := \frac{q^{1/3}}{1} \, {}_{+} \, \frac{q+q^2}{1} \, {}_{+} \, \frac{q^2+q^4}{1} \, {}_{+} \, \cdots, \qquad |q|<1, \qquad (124)$$

Thank You