DISCARDED PORTIONS OF PUBLISHED MANUSCRIPTS, AND UNPUBLISHED MANUSCRIPTS PHOTOCOPIED WITH RAMANUJAN'S LOST NOTEBOOK

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Ramanujan Receives a degree at Cambridge

Figure: Ramanujan Receiving a Degree by Research

Partial Manuscripts Originally Intended for Papers Published by Ramanujan

Discarded Work

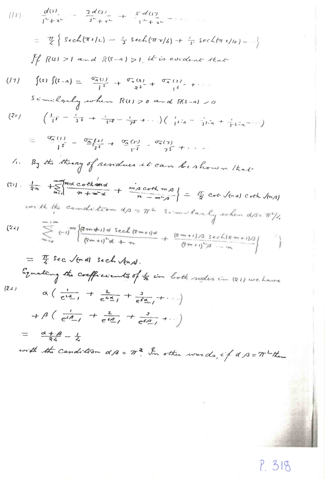
1. S. Ramanujan, *On the product* $\prod_{n=0}^{\infty}\left[1+\left(\frac{x}{a+nd}\right)^3\right]$, J. Indian Math. Soc. **7** (1915), 209–211.
2. S. Ramanujan, *Some definite integrals*, Mess. Math. **44** (1915), 10–18.
3. S. Ramanujan, *Some definite integrals connected with Gauss's sums*, Mess. Math. **44** (1915), 75–85.
4. S. Ramanujan, *Some definite integrals*, J. Indian Math. Soc. **11** (1915), 81–87.
5. S. Ramanujan, *On certain infinite series*, Mess. Math. **45** (1916), 11–15.
6. S. Ramanujan, *Some formulae in the analytic theory of numbers*, Mess. Math. **45** (1916), 81–84.
7. S. Ramanujan, *On certain trigonometric sums and their applications in the theory of numbers*, Trans. Cambridge Philos. Soc. **22** (1918), 259–276.

Unpublished Manuscripts

1. Three partial manuscripts on Diophantine Approximation
2. Three partial manuscripts on Fourier Analysis, Fourier Transforms, and Mellin Transforms
3. Two Partial manuscripts on Euler's constant
4. Two partial manuscripts on primes.
5. One wild manuscript on (mostly) analysis (from Ramanujan's early days in Kumbakonam?)

What We Do Not Discuss

1. Ramanujan's Unpublished Manuscript on the partition and tau functions
2. The completion of Ramanujan's paper on highly composite numbers
3. Lists (E.g., Identites for the Rogers–Ramanujan functions; Euler Products)
4. Fragments
5. Letters from Ramanujan to Hardy from nursing homes
6. The Original Lost Notebook



(18) $$\frac{d(1)}{1^2+x^2} - \frac{3\,d(3)}{3^2+x^2} + \frac{5\,d(5)}{5^2+x^2} - \cdots$$

$$= \frac{\pi}{4}\left\{\operatorname{sech}(\pi x/2) - \frac{1}{3}\operatorname{sech}(\pi x/6) + \frac{1}{5}\operatorname{sech}(\pi x/10) - \cdots\right\}$$

If $R(s) > 1$ and $R(s-n) > 1$, it is evident that

(19) $$\zeta(s)\,\zeta(s-n) = \frac{\sigma_n(1)}{1^s} + \frac{\sigma_n(2)}{2^s} + \frac{\sigma_n(3)}{3^s} + \cdots$$

Similarly when $R(s) > 0$ and $R(s-n) > 0$

(20) $$\left(\frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots\right)\left(\frac{1}{1^{s-n}} - \frac{1}{3^{s-n}} + \frac{1}{5^{s-n}} - \cdots\right)$$

$$= \frac{\sigma_n(1)}{1^s} - \frac{\sigma_n(3)}{3^s} + \frac{\sigma_n(5)}{5^s} - \frac{\sigma_n(7)}{7^s} + \cdots$$

4. By the theory of residues it can be shown that

(21). $$\frac{1}{2n} + \sum_{m=1}^{m=\infty}\left\{\frac{m\alpha\coth m\alpha}{n+m^2\alpha} + \frac{m\beta\coth m\beta}{n-m^2\beta}\right\} = \frac{\pi}{2}\cot\sqrt{(n\alpha)}\coth\sqrt{(n\beta)}$$

with the condition $\alpha\beta = \pi^2$. Similarly when $\alpha\beta = \pi^2/4$,

(22) $$\sum_{m=0}^{m=\infty}(-1)^m\left\{\frac{(2m+1)\alpha\operatorname{sech}(2m+1)\alpha}{(2m+1)^2\alpha+n} + \frac{(2m+1)\beta\operatorname{sech}(2m+1)\beta}{(2m+1)^2\beta-n}\right\}$$

$$= \frac{\pi}{4}\sec\sqrt{(n\alpha)}\operatorname{sech}\sqrt{(n\beta)}.$$

Equating the coefficients of $\frac{1}{n}$ in both sides in (21) we have

(23) $$\alpha\left(\frac{1}{e^{2\alpha}-1} + \frac{2}{e^{4\alpha}-1} + \frac{3}{e^{6\alpha}-1} + \cdots\right)$$

$$+\beta\left(\frac{1}{e^{2\beta}-1} + \frac{2}{e^{4\beta}-1} + \frac{3}{e^{6\beta}-1} + \cdots\right)$$

$$= \frac{\alpha+\beta}{24} - \frac{1}{4}$$

with the condition $\alpha\beta = \pi^2$. In other words, if $\alpha\beta = \pi^2$ then

P. 318

Figure: Page 318

Pages 318–321

S. Ramanujan, *Some formulae in the analytic theory of numbers*, Mess. Math. **45** (1916), 81–84.

Pages 318–321

S. Ramanujan, *Some formulae in the analytic theory of numbers*, Mess. Math. **45** (1916), 81–84.

Entry (p. 318, formula (21); Corrected Version)

If α *and* β *are positive numbers such that* $\alpha\beta = \pi^2$, *then*

$$\frac{\pi}{2}\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta}) = \frac{1}{2w} + \frac{1}{2}\log\frac{\beta}{\alpha} + \sum_{m=1}^{\infty}\left\{\frac{m\alpha\coth(m\alpha)}{w+m^2\alpha} + \frac{m\beta\coth(m\beta)}{w-m^2\beta}\right\}.$$

Pages 318–321

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The expression $\frac{1}{2}\log(\beta/\alpha)$ does not appear in the partial manuscript.

Pages 318–321

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The expression $\frac{1}{2}\log(\beta/\alpha)$ does not appear in the partial manuscript.

"By the theory of residues it can be shown that"

A Corollary That Does Not Follow

Entry (p. 320, formula (29))

If α *and* β *are positive numbers such that* $\alpha\beta = \pi^2$*, and if* $\sigma_k(m) = \sum_{d|m} d^k$*, then*

$$\sum_{m=1}^{\infty} \frac{1}{m(e^{2m\alpha}-1)} - \sum_{m=1}^{\infty} \frac{1}{m(e^{2m\beta}-1)}$$

$$= \sum_{m=1}^{\infty} \sigma_{-1}(m) e^{-2m\alpha} - \sum_{m=1}^{\infty} \sigma_{-1}(m) e^{-2m\beta} = \frac{1}{4}\log\frac{\alpha}{\beta} - \frac{\alpha-\beta}{12}.$$

Dedekind Eta Function

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}$$
$$= (q;q)_\infty = q^{-1/24}\eta(\tau), \quad q = e^{2\pi i\tau}$$

Dedekind Eta Function

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$$= (q;q)_\infty = q^{-1/24}\eta(\tau), \quad q = e^{2\pi i\tau}$$

$$\eta(-1/\tau) = \sqrt{\tau/i}\,\eta(\tau)$$

Worry about Partial Fractions?

S. Ramanujan, *On the product* $\prod_{n=0}^{\infty}\left[1+\left(\frac{x}{a+nd}\right)^3\right]$, J. Indian Math. Soc. **7** (1915), 209–211.

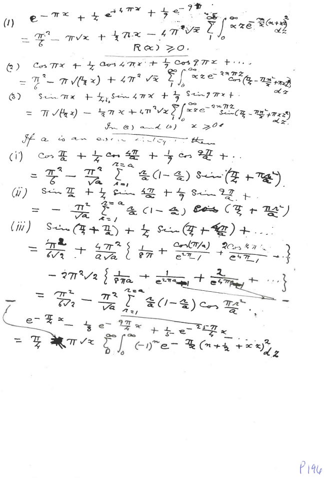
Worry about Partial Fractions?

S. Ramanujan, *On the product* $\prod_{n=0}^{\infty}\left[1+\left(\frac{x}{a+nd}\right)^3\right]$, J. Indian Math. Soc. **7** (1915), 209–211.

"It can easily be shown by the theory of residues, that

$$\frac{1}{16\pi\alpha^4}+\sum_{n=1}^{\infty}\frac{n\coth n\pi}{n^4+4\alpha^4}=\frac{\pi}{8\alpha^2}\cdot\frac{\cosh 2\pi\alpha+\cos 2\pi\alpha}{\cosh 2\pi\alpha-\cos 2\pi\alpha}."$$

Page 196 From the Lost Notebook



(1) $e^{-\pi x} + \frac{1}{4} e^{-4\pi x} + \frac{1}{9} e^{-9\pi} \cdots$

$= \frac{\pi^2}{6} - \pi\sqrt{x} + \frac{1}{2}\pi x - 4\pi^2\sqrt{x} \sum_1^\infty \int_0^\infty x z e^{-\frac{\pi}{x}(n+z)^2} dz$

$R(x) \geq 0.$

(2) $\cos \pi x + \frac{1}{4}\cos 4\pi x + \frac{1}{9}\cos 9\pi x + \cdots$

$= \frac{\pi^2}{6} - \pi\sqrt{(\frac{1}{2}x)} + 4\pi^2\sqrt{x} \sum_1^\infty \int_0^\infty x z e^{-2n\pi z} \cos(\frac{\pi}{4} - \frac{\pi n^2}{x} + \pi x z^2) dz$

(3) $\sin \pi x + \frac{1}{4}\sin 4\pi x + \frac{1}{9}\sin 9\pi x + \cdot$

$= \pi\sqrt{(\frac{1}{2}x)} - \frac{1}{2}\pi x + 4\pi^2\sqrt{x} \sum_1^\infty \int_0^\infty x z e^{-2n\pi z} \sin(\frac{\pi}{4} - \frac{\pi n^2}{x} + \pi x z^2) dz.$

In (2) and (3) $x \geq 0$

If a is an even integer, then

(i) $\cos\frac{\pi}{a} + \frac{1}{4}\cos\frac{4\pi}{a} + \frac{1}{9}\cos\frac{9\pi}{a} + \cdots$

$= \frac{\pi^2}{6} - \frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{r=a} \frac{r}{a}(1 - \frac{r}{a}) \sin(\frac{\pi}{4} + \frac{\pi r^2}{a}).$

(ii) $\sin\frac{\pi}{a} + \frac{1}{4}\sin\frac{4\pi}{a} + \frac{1}{9}\sin\frac{9\pi}{a} + \cdot$

$= -\frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{r=a} \frac{r}{a}(1 - \frac{r}{a}) \cos(\frac{\pi}{4} + \frac{\pi r^2}{a})$

(iii) $\sin(\frac{\pi}{4} + \frac{\pi}{a}) + \frac{1}{4}\sin(\frac{\pi}{4} + \frac{4\pi}{a}) + \cdots$

$= \frac{\pi^2}{6\sqrt{2}} + \frac{4\pi^2}{a\sqrt{a}} \left\{ \frac{1}{8\pi} + \frac{\cos(\pi/a)}{e^{2\pi}-1} + \frac{2\cos 4\pi/a}{e^{4\pi}-1} + \cdots \right\}$

$- 2\pi^2\sqrt{2} \left\{ \frac{1}{8\pi a} + \frac{1}{e^{2\pi a}-1} + \frac{2}{e^{4\pi a}-1} + \cdots \right\}$

$= \frac{\pi^2}{6\sqrt{2}} - \frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{r=a} \frac{r}{a}(1 - \frac{r}{a}) \cos\frac{\pi r^2}{a}.$

$e^{-\frac{\pi}{4}x} - \frac{1}{3} e^{-\frac{9\pi}{4}x} + \frac{1}{5} e^{-\frac{25\pi}{4}x} - \cdots$

$= \frac{\pi}{4}$ [illegible] $\pi\sqrt{x} \sum_0^\infty \int_0^\infty (-1)^n e^{-\frac{\pi}{x}(n + \frac{1}{2} + x z)^2} dz$

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Figure: Page 196

Very Interesting Formulas on Page 196

1. S. Ramanujan, *Some definite integrals connected with Gauss's sums*, Mess. Math. **44** (1915), 75–85.
2. S. Ramanujan, *Some definite integrals*, J. Indian Math. Soc. **11** (1915), 81–87.

Very Interesting Formulas on Page 196

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Entry (p. 196)

Let a be an even positive integer. Then

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi n^2}{a}\right)}{n^2} = \frac{\pi^2}{6} - \frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{a} \frac{r}{a}\left(1 - \frac{r}{a}\right) \sin\left(\frac{\pi}{4} + \frac{\pi r^2}{a}\right),$$

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi n^2}{a}\right)}{n^2} = -\frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{a} \frac{r}{a}\left(1 - \frac{r}{a}\right) \cos\left(\frac{\pi}{4} + \frac{\pi r^2}{a}\right).$$

Page 196 Continued

B. C. Berndt, H. H. Chan, and Y. Tanigawa, *Two Dirichlet series evaluations found on page* 196 *of Ramanujan's lost notebook*, Math. Proc. Cambridge Philos. Soc., to appear.

Page 196 Continued

B. C. Berndt, H. H. Chan, and Y. Tanigawa, *Two Dirichlet series evaluations found on page* 196 *of Ramanujan's lost notebook*, Math. Proc. Cambridge Philos. Soc., to appear.

If we combine the different evaluations, we obtain the identities

$$\frac{\pi^2}{6a^2}+\frac{\pi^2\cos(\pi a/4)}{2a^2}+\frac{\pi^2}{a^2}\sum_{j=1}^{\frac{1}{2}a-1}\cos\left(\frac{\pi j^2}{a}\right)\csc^2\left(\frac{\pi j}{a}\right)$$

$$=\frac{\pi^2}{6}-\frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\sin\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right),$$

$$\frac{\pi^2\sin(\pi a/4)}{2a^2}+\frac{\pi^2}{a^2}\sum_{j=1}^{\frac{1}{2}a-1}\sin\left(\frac{\pi j^2}{a}\right)\csc^2\left(\frac{\pi j}{a}\right)$$

$$=-\frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\cos\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right).$$

Page 196, Continued

Why are these formulas interesting?

Page 196, Continued

Why are these formulas interesting?

1. On the left side we have "almost" a Gauss sum.
   There is an "extra" factor

$$\csc^2\left(\frac{\pi j}{a}\right)$$

2. On the right side we have "almost" a Gauss sum. We have an "extra" factor of a polynomial of degree 2.
3. Have you ever seen a finite trigonometric identity involving polynomials in the summands.
4. The polynomial is "almost" the second Bernoulli polynomial, $B_2(x)$.

More General Evaluations

Theorem

If r and a are even positive integers, then

$$\sum_{n=1}^{\infty} \frac{\cos\left(\pi n^2/a\right)}{n^r} = \frac{(-1)^{1+r/2} 2^{r-1} \pi^r}{r! \sqrt{a}} \sum_{m=0}^{a-1} B_r\left(\frac{m}{a}\right) \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right)$$

and

$$\sum_{n=1}^{\infty} \frac{\sin\left(\pi n^2/a\right)}{n^r} = \frac{(-1)^{1+r/2} 2^{r-1} \pi^r}{r! \sqrt{a}} \sum_{m=0}^{a-1} B_r\left(\frac{m}{a}\right) \cos\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right).$$

Entry (p. 196)

If a is an even positive integer, then

$$\frac{4\pi^2}{a^{3/2}}\left\{\frac{1}{8\pi}+\sum_{n=1}^{\infty}\frac{n\cos(\pi n^2/a)}{e^{2n\pi}-1}\right\}-2^{3/2}\pi^2\left\{\frac{1}{8\pi a}+\sum_{n=1}^{\infty}\frac{n}{e^{2n\pi a}-1}\right\}$$
$$=-\frac{\pi^2}{a^{5/2}}\sum_{r=1}^{a}r(a-r)\cos\left(\frac{\pi r^2}{a}\right).$$

Definite Integrals on Pages 190, 191

If $\alpha\beta = \pi$,

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-(\alpha x)^2}}{\cosh \pi x}dx = \sqrt{\beta}\int_0^\infty \frac{e^{-(\beta x)^2}}{\cosh \pi x}dx$$

Definite Integrals on Pages 190, 191

If $\alpha\beta = \pi$,

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-(\alpha x)^2}}{\cosh \pi x}dx = \sqrt{\beta}\int_0^\infty \frac{e^{-(\beta x)^2}}{\cosh \pi x}dx$$

If $\alpha\beta = 2\pi$,

$$\sqrt{\alpha}\int_0^\infty \frac{\cosh \pi x}{\cosh 2\pi x}e^{-(\alpha x)^2}dx = \sqrt{\beta}\int_0^\infty \frac{\cosh \pi x}{\cosh 2\pi x}e^{-(\beta x)^2}dx$$

Definite Integrals on Pages 190, 191

If $\alpha\beta = \pi$,

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$$\sum_{n=0}^\infty \frac{(-1)^{n(n-1)/2}}{(2n+1)^s}$$

Definite Integrals on Pages 190, 191

If $\alpha\beta = \pi$,

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-(\alpha x)^2}}{\cosh \pi x}dx = \sqrt{\beta}\int_0^\infty \frac{e^{-(\beta x)^2}}{\cosh \pi x}dx$$

If $\alpha\beta = 2\pi$,

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$$\sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2}}{(2n+1)^s}$$

E. C. Titchmarsh, *Theory of Fourier Integrals*

E. C. Titchmarsh

Figure: E. C. Titchmarsh

Pages 270, 271

S. Ramanujan, *On certain trigonometric sums and their applications in the theory of numbers*, Trans. Cambridge Philos. Soc. **22** (1918), 259–276.

Pages 270, 271

S. Ramanujan, *On certain trigonometric sums and their applications in the theory of numbers*, Trans. Cambridge Philos. Soc. **22** (1918), 259–276.

Entry (pp. 270, 271)

If $s > 2$, *then*

$$\sum_{n=1}^{\infty} \sigma_{s-1}(n) e^{-2\pi n} = \frac{\Gamma(s)}{(2\pi)^s}\zeta(s)\left\{1 + 2\cos\frac{\pi s}{4}\sum \frac{\cos\left(s\tan^{-1}\frac{\mu-\nu}{\mu+\nu}\right)}{(\mu^2+\nu^2)^{\frac{1}{2}s}}\right\}$$

$$= \frac{\Gamma(s)}{(2\pi)^s}\zeta(s)\left\{1 + 2\cos\frac{\pi s}{4}\left(\frac{1}{2^{\frac{1}{2}s}} + \frac{2\cos\left(s\tan^{-1}\frac{1}{3}\right)}{5^{\frac{1}{2}s}} + \frac{2\cos\left(s\tan^{-1}\frac{1}{2}\right)}{10^{\frac{1}{2}s}} + \frac{2\cos\left(s\tan^{-1}\frac{1}{5}\right)}{13^{\frac{1}{2}s}} + \cdots\right)\right\},$$

where the sum is over all coprime positive integers μ *and* ν.

Entry (Formula (9), Page 277)

$$\sum_{k=1}^{\infty} \frac{k^{n-1}}{e^{2\pi k}-1} = \frac{|B_n|}{2n} + \frac{|B_n|}{n} \cos\left(\frac{\pi n}{4}\right) \left\{ \frac{1}{2^{n/2}} + \frac{2\cos(n\tan^{-1}\frac{1}{3})}{5^{n/2}} + \frac{2\cos(n\tan^{-1}\frac{1}{3})}{5^{n/2}} + \frac{2\cos(n\tan^{-1}\frac{1}{3})}{5^{n/2}} + \cdots \right\},$$

where 2, 5, 10, 13,. . . *are sum of squares of numbers that are prime to each other.*

Entry (Formula (9), Page 277)

$$\sum_{k=1}^{\infty} \frac{k^{n-1}}{e^{2\pi k}-1} = \frac{|B_n|}{2n} + \frac{|B_n|}{n} \cos\left(\frac{\pi n}{4}\right) \left\{ \frac{1}{2^{n/2}} + \frac{2\cos(n\tan^{-1}\frac{1}{3})}{5^{n/2}} + \frac{2\cos(n\tan^{-1}\frac{1}{3})}{5^{n/2}} + \frac{2\cos(n\tan^{-1}\frac{1}{3})}{5^{n/2}} + \cdots \right\},$$

where 2, 5, 10, 13,... *are sum of squares of numbers that are prime to each other.*

$$\sum_{k=1}^{\infty} \frac{k^{4m+1}}{e^{2\pi k}-1} = \frac{B_{4m+2}}{8m+4} \qquad (n = 4m+2).$$

Entry (Formula (9), Page 277)

$$\sum_{k=1}^{\infty} \frac{k^{n-1}}{e^{2\pi k} - 1} = \frac{|B_n|}{2n} + \frac{|B_n|}{n} \cos\left(\frac{\pi n}{4}\right) \left\{ \frac{1}{2^{n/2}} + \frac{2\cos(n \tan^{-1} \frac{1}{3})}{5^{n/2}} + \frac{2\cos(n \tan^{-1} \frac{1}{3})}{5^{n/2}} + \frac{2\cos(n \tan^{-1} \frac{1}{3})}{5^{n/2}} + \cdots \right\},$$

where 2, 5, 10, 13,... are sum of squares of numbers that are prime to each other.

$$\sum_{k=1}^{\infty} \frac{k^{4m+1}}{e^{2\pi k} - 1} = \frac{B_{4m+2}}{8m+4} \qquad (n = 4m+2).$$

J. W. L. Glaisher, 1889

A. Hurwitz, equivalent result, Ph.D. thesis, 1881

B. C. Berndt and P. Bialek, *Five formulas of Ramanujan arising from Eisenstein series*, in *Number Theory*, K. Dilcher, ed., CMS Conf. Proc., vol. 15, American Mathematical Society, Providence, RI, 1995, pp. 67–86.

Page 255

B. C. Berndt and P. Bialek, *Five formulas of Ramanujan arising from Eisenstein series*, in *Number Theory*, K. Dilcher, ed., CMS Conf. Proc., vol. 15, American Mathematical Society, Providence, RI, 1995, pp. 67–86.

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B. C. Berndt and P. Pongsriiam, *Discarded Fragments from Ramanujan's Papers*, Kubilius Memorial Volume, to appear.

Entry (p. 255)

$$1^s\sigma_r(1) + 2^s\sigma_r(2) + 3^s\sigma_r(3) + \cdots + n^s\sigma_r(n)$$

lies between

$$\zeta(-s)\zeta(-r-s) + \frac{n^{1+s}}{1+s}\zeta(1-r) + \frac{n^{1+r+s}}{1+r+s}\zeta(1+r) + \frac{1}{2}n^s\zeta(-r) + \frac{1}{2}n^{r+s}\zeta(r) + \frac{n^{s+(r+1)/2}}{1-r^2} \qquad (1)$$

and

$$\zeta(-s)\zeta(-r-s) + \frac{n^{1+s}}{1+s}\zeta(1-r) + \frac{n^{1+r+s}}{1+r+s}\zeta(1+r) - \frac{n^{s+(r+1)/2}}{1-r^2} + \frac{1}{2}n^s\left\{2\zeta(1-r) - \zeta(-r)\right\} + \frac{1}{2}n^{r+s}\left\{2\zeta(1+r) - \zeta(r)\right\}. \qquad (2)$$

Entry (p. 255)

$$1^s\sigma_r(1) + 2^s\sigma_r(2) + 3^s\sigma_r(3) + \cdots + n^s\sigma_r(n)$$

lies between

$$\zeta(-s)\zeta(-r-s) + \frac{n^{1+s}}{1+s}\zeta(1-r) + \frac{n^{1+r+s}}{1+r+s}\zeta(1+r) + \frac{1}{2}n^s\zeta(-r) + \frac{1}{2}n^{r+s}\zeta(r) + \frac{n^{s+(r+1)/2}}{1-r^2} \qquad (1)$$

and

$$\zeta(-s)\zeta(-r-s) + \frac{n^{1+s}}{1+s}\zeta(1-r) + \frac{n^{1+r+s}}{1+r+s}\zeta(1+r) - \frac{n^{s+(r+1)/2}}{1-r^2} + \frac{1}{2}n^s\left\{2\zeta(1-r) - \zeta(-r)\right\} + \frac{1}{2}n^{r+s}\left\{2\zeta(1+r) - \zeta(r)\right\}. \qquad (2)$$

We need some hypotheses and make some comments.

1. The error term must be $o(1)$, as $n \to \infty$.
2. $$s + \tfrac{1}{2}r < 0, \quad s + r < 1, \quad \text{and} \quad s < 1. \tag{3}$$
3. For (1) to hold, we need either $s > 0$ or $s + r > 0$.
4. There are two "extra" terms in (2).
5. It is impossible to state an inequality in (2) without $o(1)$ term.
6. We need to estimate

$$\sum_{k \le \sqrt{n}} \left\{ \frac{n}{k} \right\} \frac{1}{k^r}$$

(and a similar sum)

Theorem

Let s and r be real numbers satisfying the inequalities (3). Then, for n sufficiently large,

$$S(s,r)=\sum_{k=1}^{n}k^{s}\sigma_{r}(k)\le\zeta(-s)\zeta(-s-r)+\frac{n^{s+1}}{s+1}\zeta(1-r)+\frac{n^{s+r+1}}{s+r+1}\zeta(r+1)+\frac{n^{s+\frac{1}{2}(r+1)}}{1-r^{2}}+\frac{n^{s}}{2}\zeta(-r)+\frac{n^{s+r}}{2}\zeta(r),$$

provided that either $s>0$ or $s+r>0$, and

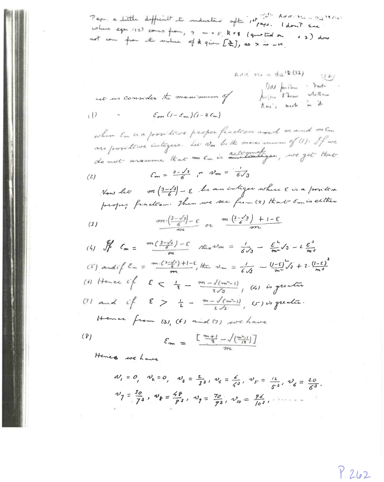
$$S(s,r)\ge\zeta(-s)\zeta(-s-r)+\frac{n^{s+1}}{s+1}\zeta(1-r)+\frac{n^{s+r+1}}{s+r+1}\zeta(r+1)-\frac{n^{s+\frac{1}{2}(r+1)}}{1-r^{2}}-\frac{n^{s}}{2}\zeta(-r)-\frac{n^{s+r}}{2}\zeta(r)+o(1).$$

Partial Manuscripts Never Completed by Ramanujan

G. H. Hardy

Figure: G. H. Hardy

Page 262 From the Lost Notebook



Paper a little difficult to understand after 1st page. I don't see where eqn (12) comes from, & m = 5, k = 8 (quoted on p. 3) does not come from the value of k given $[\frac{3}{n}]$, as x is $-ve$.

Add. Ms. a 94[18] (12)

Odd problem. Don't profess I know whether there's much in it.

Let us consider the maximum of

(1) $\varepsilon_m(1-\varepsilon_m)(1-2\varepsilon_m)$

when ε_m is a positive proper fraction and m and $m\varepsilon_m$ are positive integers. Let v_m be the maximum of (1). If we do not assume that $m\varepsilon_m$ is rational, we get that

(2) $\varepsilon_m = \frac{3-\sqrt{3}}{6}$, & $v_m = \frac{1}{6\sqrt{3}}$

Now let $m\left(\frac{3-\sqrt{3}}{6}\right) - \varepsilon$ be an integer where ε is a positive proper fraction. Then we see from (2) that ε_m is either

(3) $\dfrac{m\cdot\left(\frac{3-\sqrt{3}}{6}\right) - \varepsilon}{m}$ or $\dfrac{m\left(\frac{3-\sqrt{3}}{6}\right) + 1 - \varepsilon}{m}$

(4) If $\varepsilon_m = \dfrac{m\left(\frac{3-\sqrt{3}}{6}\right) - \varepsilon}{m}$ then $v_m = \frac{1}{6\sqrt{3}} - \frac{\varepsilon^2}{m^2}\sqrt{3} - 2\frac{\varepsilon^3}{m^3}$

(5) and if $\varepsilon_m = \dfrac{m\cdot\left(\frac{3-\sqrt{3}}{6}\right) + 1 - \varepsilon}{m}$, then $v_m = \frac{1}{6\sqrt{3}} - \frac{(1-\varepsilon)^2}{m^2}\sqrt{3} + 2\cdot\frac{(1-\varepsilon)^3}{m^3}$

(6) Hence if $\varepsilon < \frac{1}{2} - \frac{m - \sqrt{(m^2-1)}}{2\sqrt{3}}$, (4) is greater

(7) and if $\varepsilon > \frac{1}{2} - \frac{m - \sqrt{(m^2-1)}}{2\sqrt{3}}$, (5) is greater.

Hence from (3), (6) and (7) we have

(8) $$\varepsilon_m = \frac{\left[\frac{m+1}{2} - \sqrt{\left(\frac{m^2-1}{12}\right)}\right]}{m}$$

Hence we have

$v_1 = 0$, $v_2 = 0$, $v_3 = \frac{2}{3^3}$, $v_4 = \frac{6}{4^3}$, $v_5 = \frac{12}{5^3}$, $v_6 = \frac{20}{6^3}$,

$v_7 = \frac{30}{7^3}$, $v_8 = \frac{48}{8^3}$, $v_9 = \frac{70}{9^3}$, $v_{10} = \frac{96}{10^3}$, ...

P. 262

Figure: Page 262

Pages 262–265

"Paper a little difficult to understand after the first page."
(Gertrude Stanley)

Pages 262–265

"Paper a little difficult to understand after the first page." (Gertrude Stanley)

"Odd problem. I don't profess to know whether there's much to it." (G. H. Hardy)

Pages 262–265

"Paper a little difficult to understand after the first page." (Gertrude Stanley)

"Odd problem. I don't profess to know whether there's much to it." (G. H. Hardy)

"Let us consider the maximum of

$$\epsilon_m(1 - \epsilon_m)(1 - 2\epsilon_m) \qquad (4)$$

when ϵ_m is a positive proper fraction and m and $m\epsilon_m$ are positive integers. Let v_m be the maximum of (4)."

Pages 262–265

Theorem

For all values of m,

$$v_m \geq \frac{m^2-4}{6m^3}\sqrt{\frac{m^2-1}{3}}.$$

and

$$v_m \leq \frac{(m^2-1)}{6m^3}\sqrt{\frac{m^2+2}{3}},$$

with equality holding above when

$$\epsilon_m = \frac{1}{m}\left(\frac{m}{2}-\sqrt{\frac{m^2+2}{12}}\right). \tag{5}$$

Ramanujan seeks to determine the maximum value of k in order that

$$v_m = v_{2m} = v_{3m} = \cdots = v_{km}. \tag{6}$$

Theorem

As in (5), consider only those values of m for which

$$\epsilon_m = \frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2+2}{12}}\right)$$

is a rational number. Let k be the maximum value such that (6) holds. Then

$$k \ngtr \left[\frac{x}{m}\right] = \sqrt{3m^2+6} - 1,$$

where

$$\frac{1}{x}\left(\frac{x-1}{2} - \sqrt{\frac{x^2-1}{12}}\right) = \frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2+2}{12}}\right) = \epsilon_m.$$

Problem 784, J. Indian Math. Soc.

S. Ramanujan, *Question* 784, J. Indian Math. Soc. **8** (1916), 159.

Problem 784, J. Indian Math. Soc.

S. Ramanujan, *Question* 784, J. Indian Math. Soc. **8** (1916), 159. If $F(x)$ denotes the fractional part of x (e.g. $F(\pi) = 0.14159\ldots$), and if N is a positive integer, shew that

$$\liminf_{N\to\infty} NF(N\sqrt{2}) = \frac{1}{2\sqrt{2}}, \qquad \liminf_{N\to\infty} NF(N\sqrt{3}) = \frac{1}{\sqrt{3}},$$

$$\liminf_{N\to\infty} NF(N\sqrt{5}) = \frac{1}{2\sqrt{5}}, \qquad \liminf_{N\to\infty} NF(N\sqrt{6}) = \frac{1}{\sqrt{6}},$$

$$\liminf_{N\to\infty} NF(N\sqrt{7}) = \frac{3}{2\sqrt{7}},$$

$$\liminf_{N\to\infty} N(\log N)^{1-p} F(Ne^{2/n}) = 0, \tag{7}$$

where n is any integer and p is any positive number; shew further that in (7) p cannot be zero.

Problem 784, J. Indian Math. Soc.

A. A. Krisnaswami Aiyangar, *Partial solution to Question* 784, J. Indian Math. Soc. **18** (1929–30), 214–217.

Problem 784, J. Indian Math. Soc.

A. A. Krisnaswami Aiyangar, *Partial solution to Question* 784, J. Indian Math. Soc. **18** (1929–30), 214–217.

T. Vijayaraghavan and G.N. Watson, *Solution to Question* 784, J. Indian Math. Soc. **19** (1931), 12–23.

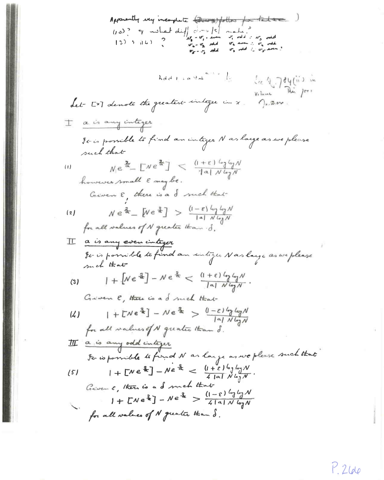
Problem 784, J. Indian Math. Soc.

A. A. Krisnaswami Aiyangar, *Partial solution to Question* 784, J. Indian Math. Soc. **18** (1929–30), 214–217.

T. Vijayaraghavan and G.N. Watson, *Solution to Question* 784, J. Indian Math. Soc. **19** (1931), 12–23.

At the top of page 266, Hardy writes, "See Q. 784(ii) in volume. This goes further,"

Page 266 From the Lost Notebook



Apparently very incomplete ~~(... follow for ...)~~)
(10)? of what diff does |c| make?
(13) & (16) ?

$v_3 - v_1$ even v_1 odd ∴ v_3 odd
$v_4 - v_2$ odd v_2 even ∴ v_4 odd
$v_5 - v_3$ odd v_3 odd ∴ v_5 even?

See Q. 784(ii) in Volume This Journal.

Let $[x]$ denote the greatest integer in x.

I <u>a is any integer</u>

It is possible to find an integer N as large as we please such that

(1) $$Ne^{\frac{2}{a}} - [Ne^{\frac{2}{a}}] < \frac{(1+\varepsilon)\log\log N}{|a|\, N\log N}$$

however small ε may be.

Given ε, there is a δ such that

(2) $$Ne^{\frac{2}{a}} - [Ne^{\frac{2}{a}}] > \frac{(1-\varepsilon)\log\log N}{|a|\, N\log N}$$

for all values of N greater than δ.

II <u>a is any even integer</u>

It is possible to find an integer N as large as we please such that

(3) $$1 + [Ne^{\frac{2}{a}}] - Ne^{\frac{2}{a}} < \frac{(1+\varepsilon)\log\log N}{|a|\, N\log N}.$$

Given ε, there is a δ such that

(4) $$1 + [Ne^{\frac{2}{a}}] - Ne^{\frac{2}{a}} > \frac{(1-\varepsilon)\log\log N}{|a|\, N\log N}$$

for all values of N greater than δ.

III <u>a is any odd integer</u>

It is possible to find N as large as we please such that

(5) $$1 + [Ne^{\frac{2}{a}}] - Ne^{\frac{2}{a}} < \frac{(1+\varepsilon)\log\log N}{4|a|\, N\log N}.$$

Given ε, there is a δ such that

$$1 + [Ne^{\frac{2}{a}}] - Ne^{\frac{2}{a}} > \frac{(1-\varepsilon)\log\log N}{4|a|\, N\log N}$$

for all values of N greater than δ.

P. 266

Figure: Page 266

Pages 266, 267

Entry

If a is any odd integer and $\epsilon > 0$ is given, then there exist infinitely many positive integers N such that

$$1 + [Ne^{2/a}] - Ne^{2/a} < \frac{(1+\epsilon)\log\log N}{4|a|N\log N}.$$

Furthermore, given $\epsilon > 0$, for all positive integers N sufficiently large,

$$1 + [Ne^{2/a}] - Ne^{2/a} > \frac{(1-\epsilon)\log\log N}{4|a|N\log N}.$$

Pages 266, 267

Entry

If a is any odd integer and $\epsilon > 0$ is given, then there exist infinitely many positive integers N such that

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Furthermore, given $\epsilon > 0$, for all positive integers N sufficiently large,

$$1 + [Ne^{2/a}] - Ne^{2/a} > \frac{(1-\epsilon)\log\log N}{4|a|N\log N}.$$

C. S. Davis, *Rational approximation to e*,
J. Austral. Math. Soc. **25** (1978), 497–502.

Formulation Due to Davis

Theorem

Let $a = \pm 2/t$*, where* t *is a positive integer, and set*

$$c = \begin{cases} 1/t, & \text{if } t \text{ is even,} \\ 1/(4t), & \text{if } t \text{ is odd.} \end{cases}$$

Then, for each $\epsilon > 0$*, the inequality*

$$\left| e^a - \frac{p}{q} \right| < (c + \epsilon) \frac{\log \log q}{q^2 \log q}$$

has an infinity of solutions in integers p, q*. Furthermore, there exists a number* q'*, depending only on* ϵ *and* t*, such that, for all integers* p, q*, with* $q \geq q'$*.*

$$\left| e^a - \frac{p}{q} \right| > (c - \epsilon) \frac{\log \log q}{q^2 \log q}.$$

Sondow's Conjecture

Theorem

Almost all partial sums of the Taylor series for e are not convergents to the continued fraction of e.

Sondow's Conjecture

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Almost all partial sums of the Taylor series for e are not convergents to the continued fraction of e.

J. Sondow and K. Schalm, *Which partial sums of the Taylor series for e are convergents to e? (and a link to the primes* 2, 5 13, 37, 463). Part II. in *Gems in Experimental Mathematics*, Contemp. Math., vol. 517, American Mathematical Society, Providence, RI, 2010, pp. 349-363.

Sondow's Conjecture

Theorem

Almost all partial sums of the Taylor series for e are not convergents to the continued fraction of e.

J. Sondow and K. Schalm, *Which partial sums of the Taylor series for e are convergents to e? (and a link to the primes* 2, 5 13, 37, 463). Part II. in *Gems in Experimental Mathematics*, Contemp. Math., vol. 517, American Mathematical Society, Providence, RI, 2010, pp. 349-363.

Sondow's Conjecture. Only two partial sums of the Taylor series for e coalesce with partial quotients of the continued fraction for e.

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Theorem

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J. Sondow and K. Schalm, *Which partial sums of the Taylor series for e are convergents to e? (and a link to the primes* 2, 5 13, 37, 463). Part II. in *Gems in Experimental Mathematics*, Contemp. Math., vol. 517, American Mathematical Society, Providence, RI, 2010, pp. 349-363.

Sondow's Conjecture. Only two partial sums of the Taylor series for e coalesce with partial quotients of the continued fraction for e.

$$\langle 2,1,2,1,1,4,1,1,6,1,1,8,1,1,\ldots\rangle = 2+\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{1}+\frac{1}{4}+\cdots.$$

Sondow's Conjecture, Cont.

Theorem

Fix a nonzero integer a. If we randomly choose one of the first n convergents to the continued fraction of $e^{2/a}$, the probability that this convergent is also a partial sum of the Taylor series of $e^{2/a}$ is

$$O_a\left(\frac{\log n}{n}\right).$$

Sondow's Conjecture, Cont.

Theorem

Fix a nonzero integer a. If we randomly choose one of the first n convergents to the continued fraction of $e^{2/a}$, the probability that this convergent is also a partial sum of the Taylor series of $e^{2/a}$ is

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Theorem

Sondow's Conjecture is true.

Sondow's Conjecture, Cont.

Theorem

Fix a nonzero integer a. If we randomly choose one of the first n convergents to the continued fraction of $e^{2/a}$, the probability that this convergent is also a partial sum of the Taylor series of $e^{2/a}$ is

$$O_a\left(\frac{\log n}{n}\right).$$

Theorem

Sondow's Conjecture is true.

B. C. Berndt, S. Kim, and A. Zaharescu, *Diophantine Approximation of the Exponential Function and Sondow's Conjecture*, submitted.

A Transformation Formula, Notation

$$\xi(s) := (s-1)\pi^{-\frac{1}{2}s}\Gamma(1+\tfrac{1}{2}s)\zeta(s).$$

Then Riemann's Ξ-function is defined by

$$\Xi(t) := \xi(\tfrac{1}{2}+it).$$

A Transformation Formula, Notation

$$\xi(s) := (s-1)\pi^{-\frac{1}{2}s}\Gamma(1+\tfrac{1}{2}s)\zeta(s).$$

Then Riemann's Ξ-function is defined by

$$\Xi(t) := \xi(\tfrac{1}{2}+it).$$

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$$

A Transformation Formula

Entry

Define

$$\phi(x) := \psi(x) + \frac{1}{2x} - \log x. \tag{8}$$

If α *and* β *are positive numbers such that* $\alpha\beta = 1$*, then*

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\phi(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\phi(n\beta)\right\}$$

$$= -\frac{1}{\pi^{3/2}}\int_0^\infty \left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt, \tag{9}$$

where γ *denotes Euler's constant and* $\Xi(x)$ *denotes Riemann's* Ξ*-function.*

Remarks on this Transformation Formula

1. Ramanujan writes that it "can be deduced from"

Entry

If $n > 0$,

$$\int_0^\infty (\psi(1+x) - \log x)\cos(2\pi nx)dx = \frac{1}{2}(\psi(1+n) - \log n)\,. \tag{10}$$

2. He probably used the Poisson summation formula.
3. The Poisson summation formula could only be used to prove the first equality.
4. The first equality in (8) established by Guinand in 1947. "This formula also seems to have been overlooked."

Remarks on this Transformation Formula

1. "Professor T. A. Brown tells me that he proved the self-reciprocal property of $\psi(1+x) - \log x$ some years ago, and that he communicated the result to the late Professor G. H. Hardy. Professor Hardy said that the result was also given in a progress report to the University of Madras by S. Ramanujan, but was not published elsewhere."
2. For $|\arg z| < \pi$, as $z \to \infty$,

$$\psi(z) \sim \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots .$$

3. Two proofs by BCB and Atul Dixit.
4. Dixit has found two further proofs, generalizations, and analogues.

A. P. Guinand

Figure: A. P. Guinand

Generalization Due to Dixit

Theorem

Let $\zeta(z,a)$ *denote the Hurwitz zeta function defined for* $a>0$ *and* $\text{Re } z > 1$ *by*

$$\zeta(z,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^z}.$$

If α *and* β *are positive numbers such that* $\alpha\beta = 1$, *then for* $\text{Re } z > 2$ *and* $1 < c < \text{Re } z - 1$,

$$\alpha^{-z/2} \sum_{k=1}^{\infty} \zeta\left(z, 1+\frac{k}{\alpha}\right) = \beta^{-z/2} \sum_{k=1}^{\infty} \zeta\left(z, 1+\frac{k}{\beta}\right)$$

$$= \frac{\alpha^{z/2}}{2\pi i \Gamma(z)} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\, ds$$

Generalization Due to Dixit, Cont.

Theorem

$$= \frac{8(4\pi)^{(z-4)/2}}{\Gamma(z)} \int_0^\infty \Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right) \times \Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt,$$

where $\Xi(t)$ is the Riemann Ξ-function.

Two Partial Manuscripts on Euler's Constant

$$\gamma = 0.57721566490153286060651209008240243104215933593992\ldots$$

$\gamma = 0.577215664901532860606512090082402431042159335939992\ldots$

Entry (p. 274)

Let p, q, and r be positive. Then

$$\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{rx^{q-1}}{1-x^r} \right) dx = \psi(q/r) - \psi(p) + \log r. \qquad (11)$$

$$\gamma = 0.57721566490153286060651209008240243104215933593992\ldots$$

Entry (p. 274)

Let p, q, and r be positive. Then

$$\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{rx^{q-1}}{1-x^r} \right) dx = \psi(q/r) - \psi(p) + \log r. \tag{11}$$

Entry (p. 274)

Suppose that a, b, and c are positive with $b > 1$. Then

$$\int_0^1 \left(\frac{x^{c-1}}{1-x} - \frac{bx^{bc-1}}{1-x^b} \right) \sum_{k=0}^{\infty} x^{ab^k} dx = \psi\left(\frac{a}{b} + c \right) - \log \frac{a}{b}.$$

Formulas for γ

Entry (p. 275)

We have

(a) $$\int_0^1 \frac{1}{1+x} \sum_{k=1}^{\infty} x^{2^k} \, dx = 1 - \gamma,$$

(b) $$\int_0^1 \frac{1+2x}{1+x+x^2} \sum_{k=1}^{\infty} x^{3^k} \, dx = 1 - \gamma,$$

(c) $$\int_0^1 \frac{1+\frac{1}{2}\sqrt{x}}{(1+\sqrt{x})(1+\sqrt{x}+x)} \sum_{k=1}^{\infty} x^{(3/2)^k} \, dx = 1 - \gamma.$$

Formula From Second Manuscript

Entry (p. 276)

$$\gamma = \log 2 - \sum_{n=1}^{\infty} 2n \sum_{k=\frac{3^{n-1}+1}{2}}^{\frac{3^n-1}{2}} \frac{1}{(3k)^3 - 3k}. \qquad (12)$$

Formula From Second Manuscript

Entry (p. 276)

$$\gamma = \log 2 - \sum_{n=1}^{\infty} 2n \sum_{k=\frac{3^{n-1}+1}{2}}^{\frac{3^n-1}{2}} \frac{1}{(3k)^3 - 3k}. \qquad (12)$$

B. C. Berndt and D. C. Bowman, *Ramanujan's short unpublished manuscript on integrals and series related to Euler's constant*, in *Constructive, Experimental and Nonlinear Analysis*, M. Thera, ed., American Mathematical Society, Providence, RI, 2000, pp. 19–27.

B. C. Berndt and T. Huber, *A fragment on Euler's constant in Ramanujan's lost notebook*, South East Asian J. Math. and Math. Sci. **6** (2008), 17–22.