RAMANUJAN'S LOST NOTEBOOK: HISTORY AND SURVEY

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Bhagavad Gita

Through long lapse of time,
This knowledge was lost.
But now, as you are devoted to truth,
I will reveal the supreme secret.

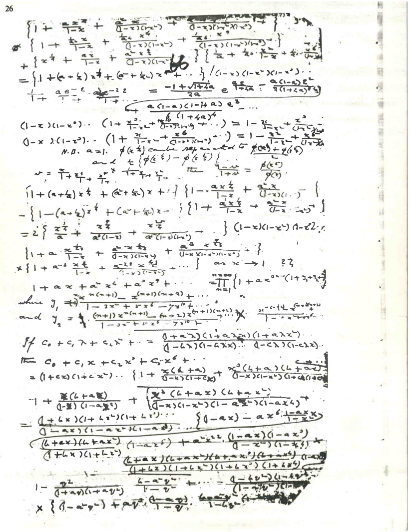
Bhagavad Gita, IV.2 & IV.3

Ramanujan's Passport Photo

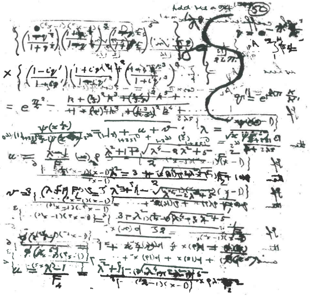
G. N. Watson

Figure: G. N. Watson

A page from Ramanujan's Lost Notebook



A page from Ramanujan's Lost Notebook



Topics in Ramanujan's Lost Notebook

q-Series

$$(a)_n := (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$$

$$(a)_\infty := (a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j), \qquad |q|<1$$

Topics in Ramanujan's Lost Notebook
q-Series

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$$(a)_\infty := (a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j), \qquad |q|<1$$

Theorem

(q-binomial theorem) *For* $|q|, |z| < 1$,

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

Theta Functions

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

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$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.$$

For any integer n,

$$(a)_n := (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

Ramanujan's ${}_1\psi_1$ summation

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Ramanujan's ${}_1\psi_1$ summation

$${}_1\psi_1(a;b;q,z) := \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(q;q)_\infty (b/a;q)_\infty (az;q)_\infty (q/(az);q)_\infty}{(b;q)_\infty (q/a;q)_\infty (z;q)_\infty (b/(az);q)_\infty}$$

where $|b/a| < |z| < 1$.

Partitions

Definition The partition function $p(n)$ is defined to be the number of ways a positive integer n can be written as a sum of positive integers.

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Example

$$p(4) = 5$$

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

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$$\begin{aligned}\frac{1}{(q;q)_\infty} &= \sum_{n_1=0}^{\infty} q^{n_1} \sum_{n_2=0}^{\infty} q^{2n_2} \cdots \sum_{n_k=0}^{\infty} q^{kn_k} \cdots \\ &= \sum_{n=0}^{\infty} p(n) q^n\end{aligned}$$

Partitions

$$
\begin{aligned}
p(5n+4) &\equiv 0 \pmod{5},\\
p(7n+5) &\equiv 0 \pmod{7},\\
p(11n+6) &\equiv 0 \pmod{11}.
\end{aligned}
$$

Partitions

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$$p(7n+5) \equiv 0 \pmod{7},$$
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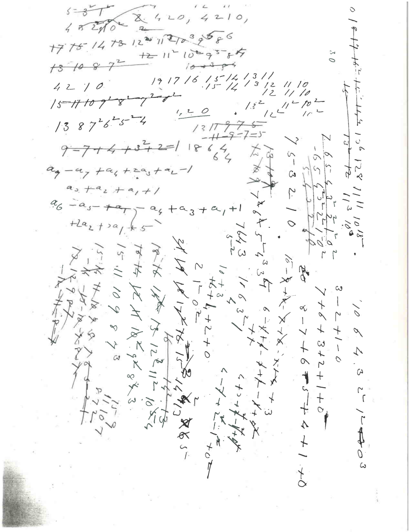
Definition

The rank of a partition is the largest part minus the number of parts.

A page from Ramanujan’s Lost Notebook



Rogers–Ramanujan functions

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \qquad \text{and} \qquad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}$$

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Rogers–Ramanujan Identities

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \qquad \text{and} \qquad H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

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Combinatorial Interpretation. The number of partitions of n into parts differing by at least 2 is equal to the number of partitions of n into parts congruent to either 1 or 4 modulo 5.

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Example.

$8 = 7 + 1 = 6 + 2 = 5 + 3$

$6 + 1 + 1 = 4 + 4 = 4 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

q-continued fractions

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots}}}}$$

$$= b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} + \cdots$$

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots}}}}$$

$$= b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} + \cdots$$

Rogers–Ramanujan continued fraction

$$R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots, \qquad |q| < 1$$

Enigmatic Continued Fraction

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \cdots, \quad |q|<1.$$

q-continued fractions

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$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \cdots, \quad |q| < 1.$$

Asymptotic Expansions of Continued Fractions

Enigmatic Continued Fraction

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Asymptotic Expansions of Continued Fractions

General Theorems

Rogers–Ramanujan Continued Fraction

$$R(1) = \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \cdots = \frac{\sqrt{5}-1}{2}$$

$$-R(-1) = \frac{1}{1} - \frac{1}{1} + \frac{1}{1} - \frac{1}{1} + \cdots = \frac{\sqrt{5}+1}{2}$$

Rogers–Ramanujan Continued Fraction

$$R(1) = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots = \frac{\sqrt{5}-1}{2}$$

$$-R(-1) = \frac{1}{1-}\,\frac{1}{1+}\,\frac{1}{1-}\,\frac{1}{1+}\cdots = \frac{\sqrt{5}+1}{2}$$

$$R(q) = q^{1/5}\frac{H(q)}{G(q)} = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

Partitions

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$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}$$

The coefficient of q^n is the number of partitions of n of even rank minus the number of partitions of n of odd rank.

Entry (p. 14)

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n} = 1 + q\sum_{n=0}^{\infty}(-1)^n (q;q)_n q^n. \qquad (1)$$

The coefficient of q^n is equal to the difference of the number of partitions of n into distinct parts with even rank, and the number of partitions of n into distinct parts with odd rank.

Ramanujan to Hardy 12 January 1920

I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary ϑ-functions.

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Sander Zwegers, Ken Ono, Kathrin Bringmann

Integrals

Complete Elliptic Integral of the First Kind

$$K(k) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

k, $0 < k < 1$, is the modulus.

Incomplete Elliptic Integral of the First Kind

$$\int_0^x \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \qquad 0 < x \leq \pi/2$$

Integrals of Dedekind Eta-functions

$$f(-q) = (q;q)_\infty = e^{-2\pi iz/24}\eta(z), \qquad q = e^{2\pi iz}, \qquad \text{Im } z > 0$$

Let

$$v := v(q) := q\frac{f^3(-q)f^3(-q^{15})}{f^3(-q^3)f^3(-q^5)}.$$

Integrals of Dedekind Eta-functions

$$f(-q) = (q;q)_\infty = e^{-2\pi iz/24}\eta(z), \qquad q = e^{2\pi iz}, \qquad \operatorname{Im} z > 0$$

Let

$$v := v(q) := q\frac{f^3(-q)f^3(-q^{15})}{f^3(-q^3)f^3(-q^5)}.$$

$$\int_0^q f(-t)f(-t^3)f(-t^5)f(-t^{15})dt$$

$$= \frac{1}{5}\int_{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{1-11v-v^2}{1+v-v^2}}\right)}^{2\tan^{-1}(1/\sqrt{5})} \frac{d\varphi}{\sqrt{1-\frac{9}{25}\sin^2\varphi}}.$$

Integral Transforms

Integral Transforms
Special Integrals

$$F_w(t) := \int_0^\infty \frac{\sin(\pi t x)}{\tanh(\pi x)} e^{-\pi w x^2} dx.$$

$$F_w(t) = -\frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} F_{1/w}(it/w).$$

Dirichlet Series and Euler Products

Further Topics

Dirichlet Series and Euler Products
Infinite Series Identities

Further Topics

Dirichlet Series and Euler Products
Infinite Series Identities
Approximations

Further Topics

Dirichlet Series and Euler Products
Infinite Series Identities
Approximations
Numerical Calculations

Further Topics

Dirichlet Series and Euler Products

Infinite Series Identities

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Diophantine Equations

Further Topics

Dirichlet Series and Euler Products
Infinite Series Identities
Approximations
Numerical Calculations
Diophantine Equations
Elementary Mathematics

Ramanujan to Hardy 16 January 1913

If

$$u = \frac{x}{1+} \frac{x^5}{1+} \frac{x^{10}}{1+} \frac{x^{15}}{1+} \frac{x^{20}}{1+} \cdots$$

and

$$v = \frac{\sqrt[5]{x}}{1+} \frac{x}{1+} \frac{x^2}{1+} \frac{x^3}{1+} \cdots,$$

then

$$v^5 = u \cdot \frac{1 - 2u + 4u^2 - 3u^3 + u^4}{1 + 3u + 4u^2 + 2u^3 + u^4}.$$

First Letter to Hardy

$$\frac{1}{1+}\frac{e^{-2\pi}}{1+}\frac{e^{-4\pi}}{1+}\frac{e^{-6\pi}}{1+}\cdots$$

$$=\left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}}.$$

First Letter to Hardy

$$\frac{1}{1+}\frac{e^{-2\pi}}{1}\;+\;\frac{e^{-4\pi}}{1}\;+\;\frac{e^{-6\pi}}{1}\;+\cdots = \left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}}.$$

$$\frac{1}{1-}\frac{e^{-\pi}}{1}\;-\;\frac{e^{-2\pi}}{1}\;-\;\frac{e^{-3\pi}}{1}\;+\cdots = \left(\sqrt{\frac{5-\sqrt{5}}{2}}-\frac{\sqrt{5}-1}{2}\right)\sqrt[5]{e^{\pi}}.$$

First Letter to Hardy

$$\frac{1}{1} + \frac{e^{-\pi\sqrt{n}}}{1} + \frac{e^{-2\pi\sqrt{n}}}{1} + \frac{e^{-3\pi\sqrt{n}}}{1} + \cdots$$

can be exactly found if n be any positive rational quantity.

Ramanujan to Hardy 27 February 1913

(1) If

$$F(x) = \frac{1}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+}\frac{x^4}{1+}\frac{x^5}{1+\cdots},$$

then

$$\left\{\frac{\sqrt{5}+1}{2} + e^{-2\alpha/5}F(e^{-2\alpha})\right\} \times \left\{\frac{\sqrt{5}+1}{2} + e^{-2\beta/5}F(e^{-2\beta})\right\} = \frac{5+\sqrt{5}}{2}$$

with the condition $\alpha\beta = \pi^2$.

Second Letter to Hardy

N.B. It is always possible to find exactly the value of

$$\frac{1}{1+}\;\frac{e^{-\pi\sqrt{n}}}{1}\;+\;\frac{e^{-2\pi\sqrt{n}}}{1}\;+\;\frac{e^{-3\pi\sqrt{n}}}{1}\;+\cdots$$

and similar continued fraction if n be any rational quantity. e.g.

$$\frac{1}{1+}\;\frac{e^{-2\pi\sqrt{5}}}{1}\;+\;\frac{e^{-4\pi\sqrt{5}}}{1}\;+\;\frac{e^{-6\pi\sqrt{5}}}{1}\;+\;\frac{e^{-8\pi\sqrt{5}}}{1}\;+\cdots$$
$$= e^{2\pi/\sqrt{5}}\left\{\frac{\sqrt{5}}{1+\sqrt[5]{5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2}-1}}-\frac{\sqrt{5}+1}{2}\right\}.$$

Second Letter to Hardy

The above theorem is a particular case of a theorem on the continued fraction

$$\frac{1}{1}+\frac{ax}{1}+\frac{ax^2}{1}+\frac{ax^3}{1}+\frac{ax^4}{1}+\frac{ax^5}{1}+\cdots,$$

The above theorem is a particular case of a theorem on the continued fraction

$$\frac{1}{1}\,+\,\frac{ax}{1}\,+\,\frac{ax^2}{1}\,+\,\frac{ax^3}{1}\,+\,\frac{ax^4}{1}\,+\,\frac{ax^5}{1}\,+\cdots,$$

which is a particular case of the continued fraction

$$\frac{1}{1}\,+\,\frac{ax}{1+bx}\,+\,\frac{ax^2}{1+bx^2}\,+\,\frac{ax^3}{1+bx^3}\,+\cdots,$$

The above theorem is a particular case of a theorem on the continued fraction

$$\frac{1}{1+}\,\frac{ax}{1+}\,\frac{ax^2}{1+}\,\frac{ax^3}{1+}\,\frac{ax^4}{1+}\,\frac{ax^5}{1+}\cdots,$$

which is a particular case of the continued fraction

$$\frac{1}{1+}\,\frac{ax}{1+bx+}\,\frac{ax^2}{1+bx^2+}\,\frac{ax^3}{1+bx^3+}\cdots,$$

which is a particular case of a general theorem on continued fractions.

Hardy to Ramanujan, 26 March 1913

Hardy to Ramanujan, 26 March 1913
(the day on which Paul Erdös was born)

Hardy to Ramanujan, 26 March 1913
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What I should like above all is a definite proof of some of your results concerning continued fractions of the type

$$\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\cdots;$$

and I am quite sure that the wisest thing you can do, in your own interests, is to let me have one as soon as possible.

Hardy to Ramanujan, 24 December 1913

If you will send me your proof written out carefully (so that it is easy to follow), I will (assuming that I agree with it–of which I have very little doubt) try to get it published for you in England. Write it in the form of a paper 'On the continued fraction

$$\frac{x}{1} + \frac{x^2}{1} + \frac{x^3}{1} + \cdots,'$$

giving a full proof of the principal and most remarkable theorem, viz. that the fraction can be expressed in finite terms when $x = e^{-\pi\sqrt{n}}$, when $\underline{n}$ is rational.

Rogers–Ramanujan Continued Fraction

$$R(q) = \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots, \qquad |q| < 1$$

Rogers–Ramanujan Continued Fraction

$$R(q) = \frac{q^{1/5}}{1} \; + \; \frac{q}{1} \; + \; \frac{q^2}{1} \; + \; \frac{q^3}{1} \; + \cdots, \qquad |q| < 1$$

$$R(1) = \frac{\sqrt{5}-1}{2}, \qquad -R(-1) = \frac{\sqrt{5}+1}{2}$$

Rogers–Ramanujan Continued Fraction

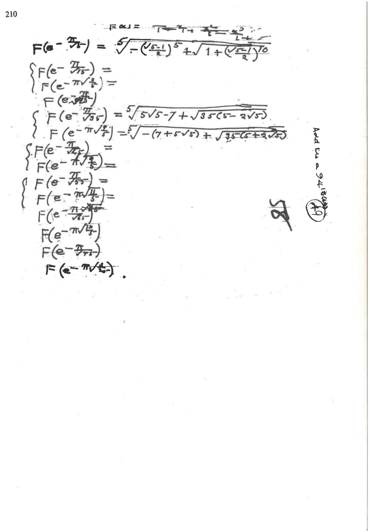
$$f(-q) = (q;q)_\infty$$

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}$$

and

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{q f^6(-q^5)}.$$

A page from Ramanujan's Lost Notebook


210

$F(e^{-\pi/\sqrt{5}}) = \sqrt[5]{-\left(\frac{\sqrt{5}-1}{2}\right)^5+\sqrt{1+\left(\frac{\sqrt{5}-1}{2}\right)^{10}}}$

$F(e^{-\pi/\sqrt{15}}) =$

$F(e^{-\pi\sqrt{3/5}}) =$

$F(e^{-\pi/\sqrt{25}})$

$F(e^{-\pi/\sqrt{35}}) = \sqrt[5]{5\sqrt{5}-7+\sqrt{35(5-2\sqrt{5})}}$

$F(e^{-\pi\sqrt{7/5}}) = \sqrt[5]{-(7+5\sqrt{5})+\sqrt{35(5+2\sqrt{5})}}$

$F(e^{-\pi/\sqrt{45}}) =$

$F(e^{-\pi\sqrt{9/5}}) =$

$F(e^{-\pi/\sqrt{55}}) =$

$F(e^{-\pi\sqrt{11/5}}) =$

$F(e^{-\pi/\sqrt{65}})$

$F(e^{-\pi\sqrt{13/5}})$

$F(e^{-\pi/\sqrt{75}})$

$F(e^{-\pi\sqrt{1/20}})$.

Some values for $R(q)$

$$S(q) := -R(-q)$$

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p. 210

$$S(e^{-\pi\sqrt{7/5}}) = \left(-5\sqrt{5} - 7 + \sqrt{35(5+2\sqrt{5})}\right)^{1/5}$$

Some values for $R(q)$

$$S(q) := -R(-q)$$

p. 210

$$S(e^{-\pi\sqrt{7/5}}) = \left(-5\sqrt{5} - 7 + \sqrt{35(5+2\sqrt{5})}\right)^{1/5}$$

p. 210 Let $a = 2\sqrt{15}$ and $b = 3\sqrt{5} - 1$. If

$$2c = \frac{a+b}{a-b}5\sqrt{5} - 11,$$

then

$$S^5(e^{-\pi\sqrt{9/5}}) = \sqrt{c^2+1} - c.$$

Class Invariants and Singular Moduli

$$\chi(q) := (-q; q^2)_\infty$$

If n is any positive rational number and $q = \exp(-\pi\sqrt{n})$,

$$G_n := 2^{-1/4} q^{-1/24} \chi(q).$$

$$\alpha = k^2, \qquad \alpha_n := \alpha(e^{-\pi\sqrt{n}})$$

is the *singular modulus*.

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24}.$$

Examples of Class Invariants

$$G_5 = \left(\frac{1+\sqrt{5}}{2}\right)^{1/4},$$

$$G_9 = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3},$$

$$G_{13} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4},$$

$$G_{69} = \left(\frac{5+\sqrt{23}}{\sqrt{2}}\right)^{1/12} \left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{1/8} \times \left(\sqrt{\frac{6+3\sqrt{3}}{4}} + \sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{1/2}$$

Examples of Class Invariants

$$
\begin{aligned}
G_{1353} = {} & (3+\sqrt{11})^{1/4}(5+3\sqrt{3})^{1/4} \\
& \times \left(\frac{11+\sqrt{123}}{2}\right)^{1/4}\left(\frac{6817+321\sqrt{451}}{4}\right)^{1/12} \\
& \times \left(\sqrt{\frac{17+3\sqrt{33}}{8}}+\sqrt{\frac{25+3\sqrt{33}}{8}}\right)^{1/2} \\
& \times \left(\sqrt{\frac{561+99\sqrt{33}}{8}}+\sqrt{\frac{569+99\sqrt{33}}{8}}\right)^{1/2}
\end{aligned}
$$

Entry (p. 207, Lost Notebook)

If

$$P = \frac{f(-\lambda^{10}q^7, -\lambda^{15}q^8) + \lambda q f(-\lambda^5 q^2, -\lambda^{20}q^{13})}{q^{1/5} f(-\lambda^{10}q^5, -\lambda^{15}q^{10})},$$

$$Q = \frac{\lambda f(-\lambda^5 q^4, -\lambda^{20}q^{11}) - \lambda^3 q f(-q, -\lambda^{25}q^{14})}{q^{-1/5} f(-\lambda^{10}q^5, -\lambda^{15}q^{10})},$$

then

Identities in Two Variables

$$
\begin{aligned}
P - Q &= 1 + \frac{f(-q^{1/5}, -\lambda q^{2/5})}{q^{1/5} f(-\lambda^{10} q^5, -\lambda^{15} q^{10})}, \\
PQ &= 1 - \frac{f(-\lambda, -\lambda^4 q^3) f(-\lambda^2 q, -\lambda^3 q^2)}{f^2(-\lambda^{10} q^5, -\lambda^{15} q^{10})}, \\
P^5 - Q^5 &= 1 + 5PQ + 5P^2 Q^2 \\
&\quad + \frac{f(-q, -\lambda^5 q^2) f^5(-\lambda^2 q, -\lambda^3 q^2)}{q\, f^6(-\lambda^{10} q^5, -\lambda^{15} q^{10})}.
\end{aligned}
$$

Identities in Two Variables

Let $\lambda = 1$.

$$P = \frac{f(-q^7, -q^8) + qf(-q^2, -q^{13})}{q^{1/5} f(-q^5)} = \frac{1}{R(q)},$$

$$Q = \frac{f(-q^4, -q^{11}) - qf(-q, -q^{14})}{q^{-1/5} f(-q^5)} = R(q).$$

Identities in Two Variables

Let $\lambda = 1$.

$$P = \frac{f(-q^7, -q^8) + qf(-q^2, -q^{13})}{q^{1/5} f(-q^5)} = \frac{1}{R(q)},$$
$$Q = \frac{f(-q^4, -q^{11}) - qf(-q, -q^{14})}{q^{-1/5} f(-q^5)} = R(q).$$

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)},$$
$$PQ = 1,$$
$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}.$$

Power Series Coefficients

$$C(q) := \frac{1}{q^{-1/5}R(q)}.$$

$$C(q) = \sum_{n=0}^{\infty} v_n q^n, \qquad |q| < 1.$$

$$\sum_{n=0}^{\infty} v_{5n} q^n = \frac{1}{(q)_\infty} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+n)/2} + q^4 \sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+49n)/2} \right).$$

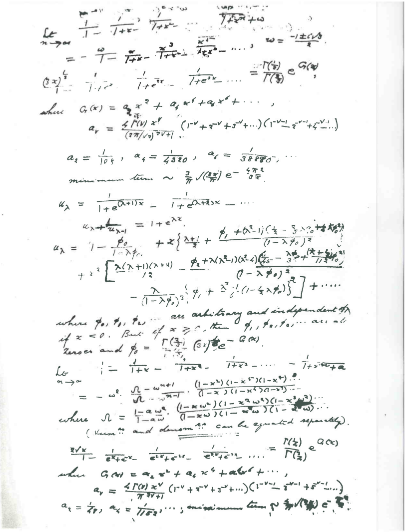
Power Series Coefficients

Corollary

We have $v_2 = v_4 = v_9 = 0$. *The remaining coefficients* v_n *satisfy the inequalities*

$$\begin{aligned} v_{5n} &> 0, \\ v_{5n+1} &> 0, \\ v_{5n+2} &< 0, \\ v_{5n+3} &< 0, \\ v_{5n+4} &< 0. \end{aligned}$$

A page from Ramanujan's Lost Notebook



where $\phi_0, \phi_1, \phi_2, \ldots$ are arbitrary and independent of λ if $x < 0$. But if $x \geq 0$, then $\phi_1, \phi_2, \phi_3, \ldots$ are all zeroes

where $\Omega = \frac{1-a\omega^2}{1-a\omega}$ (num and denom can be equated separately).

p. 45

Theorem. Let $\zeta(s)$ denote the Riemann zeta function, and let $L(s, \chi)$ denote the Dirichlet L-function associated with $\chi(n) = \left(\frac{n}{3}\right)$, where $\left(\frac{n}{3}\right)$ denotes the Legendre symbol. For each integer $n \geq 2$, let

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Then, as $x \to 0^+$,

$$\frac{(3x)^{1/3}}{1} - \frac{1}{1+e^x} - \frac{1}{1+e^{2x}} - \frac{1}{1+e^{3x}} - \cdots = \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} e^{G(x)},$$

where, as $x \to 0^+$,

$$G(x) \sim \sum_{n=1}^{\infty} a_{2n} x^{2n}.$$

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Furthermore, as $x \to 0^+$,

$$\text{the minimum value of } a_\nu x^\nu$$

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As $x \to 0^+$, if

$$R(e^{-x}) = \frac{\sqrt{5}-1}{2} e^{G(x)},$$

then, for any *large* positive number N,

$$G(x) = O(x^N).$$

The Enigmatic Continued Fraction

p. 45 If $\omega = e^{2\pi i/3}$, then for $|q| < 1$,

$$\lim_{n\to\infty} \left(\frac{1}{1} - \frac{1}{1+q} - \frac{1}{1+q^2} - \cdots - \frac{1}{1+q^n+a} \right) = -\omega^2 \left(\frac{\Omega - \omega^{n+1}}{\Omega - \omega^{n-1}} \right) \cdot \frac{(q^2;q^3)_\infty}{(q;q^3)_\infty},$$

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Bowman, McLaughlin, *Advances in Math.* **210** (2007).

Freeman Dyson
University of Illinois, Urbana
June 1, 1987

I gave thanks to Ramanujan for two things, for discovering congruence properties of partitions and for not discovering the criterion for dividing them into equal classes. That was the wonderful thing about Ramanujan. He discovered so much, and yet he left so much more in his garden for other people to discover. In the 44 years since that happy day, I have intermittently been coming back to Ramanujan's garden. Every time when I come back, I find fresh flowers blooming.