Partition Theory: Yesterday and Today

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1 Introduction

The works of Srinivasa Ramanujan - the legendary Indian mathematician of the twentieth century, made a profound impact on many areas of modern number theoretic research. For example, partitions, continued fractions, definite integrals and mock theta functions. This lecture is devoted to his crowning achievements in the theory of partitions: asymptotic properties, congruence properties and partition identities. We shall also talk about some of the advances that have occurred in these areas after Ramanujan.

The theory of partitions is an important branch of additive number theory. The concept of partition of non-negative integers also belongs to combinatorics. Partitions first appeared in a letter written by Leibnitz in 1669 to John Bernoulli, asking him if he had investigated the number of ways in which a given number can be expressed as a sum of two or more integers. The real development started with Euler (1674). It was he who first discovered the important properties of the partition function and presented them in his book " Introduction in Analysin Infinitorum". The theory has been further developed

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by many of the other great mathematicians - prominent among them are Gauss, Jacobi, Cayley, Sylvester, Hardy, Ramanujan, Schur, MacMahon, Gupta, Gordon, Andrews and Stanley. The celebrated joint work of Ramanujan with Hardy indeed revolutionized the study of partitions. Because of its great many applications in different areas like probability, statistical mechanism and particle physics, the theory of partitions has become one of the most hot research areas of the theory of numbers today.

A partition of a positive integer n is a finite non- increasing sequence of positive integers $a_1 \geq a_2 \geq \cdots \geq a_r$ such that $\sum_{i=1}^{r} a_i = n$. The a_i are called the parts or summands of the partition. We denote by $p(n)$ the number of partitions of n.

Remark 1. we observe that 0 has one partition, the empty partition, and that the empty partition has no part. We set $p(0) = 1$.

Remark 2. It is conventional to abbreviate repeated parts by the use of exponents. For example, the partitions of 4 are written as 4, 31, 2^2, 21^2, 1^4.

Remark 3. In the definition of partitions the order does not matter. 4+3 and 3+4 are the same partition of 7. Thus a partition is an unordered collection of parts. An ordered collection is called a Composition. Thus 4+3 and 3+4 are two different compositions of 7.

The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1}$$

where $\mid q \mid < 1$ and $(q;q)_n$ is a rising q-factorial defined by

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})},$$

for any constant a.

If n is a positive integer,then obviously

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

and

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots.$$

Plane partitions

A plane partition π of n is an array

$$\begin{array}{|cccc} \hline n_{1,1} & n_{1,2} & n_{1,3} & \cdots \\ n_{2,1} & n_{2,2} & n_{2,3} & \cdots \\ . & . & . & \\ . & . & . & \\ . & . & . & \end{array}$$

of positive integers which is non increasing along each row and column and such that $\Sigma n_{i,j} = n$. The entries $n_{i,j}$ are called the parts of π. A plane partition is called symmetric if $n_{i,j} = n_{j,i}$, for all i and j. If the entries of π are strictly decreasing in each column, we say that π is column strict. And if the elements of

π are strictly decreasing in each row, we call such a partition row strict. If π is both row strict and column strict, we say that π is row and column strict. Column strict plane partitions are equivalent to Young tableaux which are used in invariant theory. Plane partitions have applications in representation theory of the symmetric group, algebraic geometry and in many combinatorial problems. Plane partitions with atmost k rows are called k-line partitions. We denote by $t_k(n)$ the number of k-line partitions of n. Plane partition is a very active area of research. An extensive and readable account of work done in this area is given in [55,56]. However, in this lecture we shall only briefly touch the Ramanujan type congruence properties of $t_k(n)$.

F- partitions

A generalized Frobenius partition (or an F- partition) of n is a two rowed array of integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$, $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$, such that

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i.$$

Remark. These partitions are attributed to Frobenius because he was the first to study them in his work on group representation theory under the additional assumptions: $a_1 > a_2 > \cdots > a_r \geq 0$, $b_1 > b_2 > \cdots > b_r \geq 0$.
$\phi_k(n)$ will denote the number of F- partitions of n such that

any entry appears atmost k times on a row. An F- partition is said to be a k- coloured F- partition if in each row the parts are distinct and taken from k-copies of the non-negative integers ordered as follows:

$$0_1 < 0_2 < \cdots < 0_k < 1_1 < 1_2 < \cdots < 1_k < 2_1 < \cdots < 2_k < \cdots.$$

The notation $c\phi_k(n)$ is used to denote the number of k- coloured partitions of n. For a detailed study of F- partitions the reader is referred to [26]. In this lecture we will briefly mention some Ramanujan type congruence properties of $\phi_k(n)$ and $c\phi_k(n)$.

Partitions with "$n+t$ copies of n"

A partition with "$n+t$ copies of n," $t \geq 0$, (also called an $(n+t)$- colour partition) is a partition in which a part of size n, $n \geq 0$, can come in $(n+t)$- different colours denoted by subscripts: $n_1, n_2, \cdots, n_{n+t}$. In the part n_i, n can be zero if and only if $i \geq 1$. But in no partition are zeros permitted to repeat.
Thus, for example, the partitions of 2 with "$n+1$ copies of n" are

$$\begin{array}{cccc} 2_1, & 2_1+0_1, & 1_1+1_1, & 1_1+1_1+0_1 \\ 2_2, & 2_2+0_1, & 1_2+1_1, & 1_2+1_1+0_1 \\ 2_3, & 2_3+0_1, & 1_2+1_2, & 1_2+1_2+0_1. \end{array}$$

The weighted difference of two elements m_i and n_j, $m \geq n$ in a partition with "$n+t$ copies of n" is defined by $m-n-i-j$ and is denoted by $((m_i - n_j))$. Partitions with "$n+t$ copies of n" were used by Agarwal and Andrews [17] and Agarwal [1,2,3,4,5,6,7,9,13,15] to obtain new Rogers - Ramanujan type identities. Agarwal and Bressoud [20] and Agarwal [8,15] connected these partitions with lattice paths. Further properties of these partitions were found by Agarwal and Balasubramanian [19]. Agarwal in [10] also defined n- colour compositions. Several combinatorial properties of n- colour compositions were found in [10,12,48,49]. Anand and Agarwal [23] used n- colour partitions in studying the properties of several restricted plane partition functions. Agarwal [14] and Agarwal and Rana [21] have also used $(n+t)$- colour partitions in interpreting some mock theta functions of Ramanujan combinatorially. In fact our endeavor is to develop a complete theory for n- colour partitions parallel to the theory of classical partitions of Euler.

2 Hardy - Ramanujan - Rademacher exact formula for $p(n)$

As one many perceive, $p(n)$ grows astronomically with n. Even if a person has perfect powers of concentration and writes one partition per second, it will take him about 1,26,000 years to write all 3,972,999,029,388 partitions of 200. Ramanujan asked himself a basic question: " Can we find $p(n)$, without enumerating all the partitions of n?" This question was first answered by Hardy and Ramanujan in 1918 in their epock - making paper [39]. Their asymptotic formula for $p(n)$ can be stated as

$$p(n) = \frac{(12)^{1/2}}{(24n-1)u_n} \sum_{k=1}^{v} A_k(n)(u_n-k)exp(u_n/k)+O(n^{-1}), \quad (2.1)$$

where

$$u_n = \frac{\pi\sqrt{24n-1}}{6}, v = O\sqrt{n}$$

and

$$A_k(n) = \sum_{0\le h<k,(h,k)=1} \omega_{h,k}\; e^{-2nh\pi i/k},$$

in which $\omega_{h,k}$ a certain $24th$ root of unity. The first few terms of (2.1) give a value the integral part of which is $p(n)$ itself. However, D.H. Lehmer [44] found that the Hardy- Ramanujan series (2.1) was divergent. But H. Rademacher [50,51] proved that if in (2.1) $(u_n-k)exp(u_n/k)$ is relpaced by $(u_n-k)exp(u_n/k)+(u_n+k)exp(-u_n/k)$, then we get a convergent series for $p(n)$. The new famous Hardy - Ramanujan - Rademacher expansion for $p(n)$ is the following:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)k^{1/2}\left[\frac{d}{dx}\frac{sinh\frac{\pi}{k}[\frac{2}{3}(x-\frac{1}{24})]^{1/2}}{(x-1/24)^{1/2}}\right]_{x=n}, \quad (2.2)$$

where $A_k(n)$ are as defined earlier. Formula (2.2) is one of the most remarkable results in mathematics. It shows an interaction between an arithmetic function $p(n)$ and some techniques of calculus. It is not only a theoretical formula for $p(n)$ but also a formula which admits relatively rapid computation. For example, if we compute the first eight terms of the series for n=200, we find the result is 3,972,999,029,388.004 which is the correct value of $p(200)$ within 0.004. The 'Circle method' developed for proving formula for $p(n)$ has been useful in later developments of modular function theory.
To know more about this subject the reader is referred to Rademacher [52] and Sections P68 and P72 of LeVeque [46].

3 Ramanujan's Congruence properties of $p(n)$

By looking carefully at MacMahon's table of $p(n)$ from n=1 to 200, Ramanujan was led to conjecture the following congruences:

$$p(5m+4) \equiv 0(mod\ 5), \tag{3.1}$$

$$p(7m+5) \equiv 0(mod\ 7), \tag{3.2}$$

and

$$p(11m+6) \equiv 0(mod\ 11). \tag{3.3}$$

$$n : 4 \quad 9 \quad 14 \quad 19 \quad 24 \quad etc.$$

$$p(n) : 5 \quad 30 \quad 135 \quad 490 \quad 1575 \quad etc.$$

For elementary proofs of (3.1) and (3.2) see Ramanujan [53, 1919]. An elementary proof of (3.3) was given by Winquist [60, 1969].

There are also congruences to moduli 5^2, 7^2 and 11^2, such as

$$p(25m+24) \equiv 0(mod\ 5^2).$$

All these congruences are included in Ramanujan's famous conjecture:

If

$$\delta = 5^a 7^b 11^c \ and \ 24n \equiv 1(mod\, \delta) \ then \ p(n) \equiv 0(mod\, \delta). \tag{3.4}$$

Ramanujan proved this conjecture for 5^2, 7^2, 11^2 while Krečmar [41, 1933] proved it for 5^3, and G.N. Watson [59,1936] proved it for general 5^a. Ramanujan's conjecture held good till 1934, when S. Chowla [33, 1934], using H. Gupta's table of $p(n)$ for $n \leq 300$, found that the conjecture failed for n=243, since p(243)=133978259344888 is not divisible by 7^3 and $24.243 \equiv$

$1(mod\, 7^3)$. Watson [59,1936] modified the conjecture and proved: If $24n \equiv 1(mod \;\; 7^b)$ then $p(n) \equiv 0(mod\; 7^{[(b+2)/2]}$. The truth of Ramanujan's conjecture (3.4) was verified by Lehmer [43, 1936] for the first values of n associated with the moduli 11^3 and 11^4. Lehner [45, 1950,] proved the conjecture for 11^3. Finally, in 1967, A.O.L. Atkin [28] settled the problem by proving (3.4) for general 11^c. The full truth with regard to the conjecture can now be stated as follows:

Theorem 3.1. If $\delta = 5^a 7^b 11^c$ and $24n \equiv 1(mod\; \delta)$, then

$$p(n) \equiv 0(mod\; 5^a\; 7^{[(b+2)/2]}\; 11^c). \tag{3.5}$$

In order to obtain the combinatorial interpretations of the Ramanujan's congruences (3.1) - (3.3), Dyson [34, 1944] defined the " rank of a partition" as the largest part minus the number of parts. He conjectured the following combinatorial interpretations of the congruences (3.1) and (3.2), respectively,

Theorem 3.2. If we write $\pi \sim \pi'$ when the ranks r and r' of π and π' are congruent $(mod\; 5)$, then the equivalence classes of partitions of $5n + 4$ induced by $\sim$ are equinumerous.

Theorem 3.3. If we write $\pi \sim \pi'$ when the ranks r and r' of π and π' are congruent $(mod\; 7)$, then the equivalence classes of partitions of $7n + 5$ induced by $\sim$ are equinumerous.

Atkin and Swinnerton [29,1953] proved Theorems 3.2 and 3.3. Dyson observed that the rank did not separate the partitions of $11n + 6$ into 11 equal classes. He then conjectured that there must be some other partition statistics (which he called the "crank") that would provide a combinatorial interpretation of Ramanujan's third congruence (3.3).

Andrews and Garvan [27, 1988] defined the crank of a partition as follows:
For a partition π, let $l(\pi)$ denote the largest part of π, $\omega(\pi)$ denote the number of ones in π and $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$. The "crank" $c(\pi)$ is defined by

$$c(\pi) = \begin{cases} l(\pi) & if \quad \omega(\pi) = 0 \\ \mu(\pi) - \omega(\pi) & if \quad \omega(\pi) > 0. \end{cases}$$

They gave the following combinatorial interpretation of Ramanujan's third congruence (3.3):

Theorem 3.4. If we write $\pi \sim \pi'$ when the cranks $c(\pi)$ and $c(\pi')$ are congruent $(mod\ 11)$, then the equivalence classes of partitions of $11n + 6$ induced by $\sim$ are equinumerous.

Garvan [36,1986] gave other combinatorial interpretations of Ramanujan's congruences by using vector partitions. He defined a vector partition $\vec{\pi}$ of n as a 3-tuple (π_1, π_2, π_3), where π_1 is a partition into distinct parts and π_2 and π_3 are ordinary partitions such that the sum of the parts of the individual components of $\vec{\pi}$ is n. The rank of $\vec{\pi}$ is defined as the number of parts of π_2 minus the number of parts of π_3. Let $N_v(m, t, n)$ denote the number of vector partitions of n in which the rank is congruent to m *modulo* t. Garvan's combinatorial interpretations of (3.1) - (3.3) are

$$N_v(0,5,5n{+}4) = N_v(1,5,5n{+}4) = \cdots = N_v(4,5,5n{+}4) = \frac{p(5n+4)}{5}, \tag{3.6}$$

$$N_v(0,7,7n{+}5) = N_v(1,7,7n{+}5) = \cdots = N_v(6,7,7n{+}5) = \frac{p(7n+5)}{7}, \tag{3.7}$$

$$N_v(0,11,11n+6) = \cdots = N_v(10,11,11n+6) = \frac{p(11n+6)}{11}. \tag{3.8}$$

The study of Ramanujan type congruences is an active line of research. Agarwal and subbarao [22,1991] obtained an infinite class of Ramanujan type congruences for perfect partitions. A perfect partition of a number n is one which contains just one partition of every number less than n when repeated parts are regarded as indistinguishable. The number of perfect partitions of n is denoted by $per(n)$.
Example. per(7)=4 since there are four perfect partitions of 7, viz., 41^3, 421, $2^3 1$, 1^7

Agarwal and Subbarao proved the following:

Theorem 3.5. (Agarwal and Subbarao) For $n \geq 1$, $k \geq 2$ and q any prime,

$$per(nq^k - 1) \equiv 0(mod\ 2^{k-1}). \tag{3.9}$$

Theorem 3.5 yields infinitely many Ramanujan type congruences for perfect partitions. For example, when $k = 3$, $q = 2$ with n replaced by $n + 1$, (3.9) reduces to

$$per(8n + 7) \equiv 0(mod\ 4),$$

which is very much analogous in structure to Ramanujan's congruences (3.1) - (3.3).
Cheema and Gordon [32,1964] have obtained the following congruences for two- and three- line partitions:

Theorem 3.6. (Cheema and Gordon).

$$t_2(\nu) \equiv 0(mod\ 5), \quad if \quad \nu \equiv 3\ or\ 4(mod\ 5) \tag{3.10}$$

and

$$t_3(3\nu+2) \equiv 0(mod\ 3). \tag{3.11}$$

More congruences of this type were found by Gandhi [35,1967]. His results are given in the following theorem

Theorem 3.7. (Gandhi).

$$t_2(2\nu) \equiv t_2(2\nu+1)(mod\ 2), \tag{3.12}$$

$$t_3(3\nu) \equiv t_3(3\nu+1)(mod\ 3), \tag{3.13}$$

$$t_4(4\nu) \equiv t_4(4\nu+1) \equiv t_4(4\nu+2)(mod\ 2), \tag{3.14}$$

$$t_4(4\nu+3) \equiv 0(mod\ 2), \tag{3.15}$$

$$t_5(5\nu+1) \equiv t_5(5n+3)(mod\ 5), \tag{3.16}$$

and

$$t_5(5n+2) \equiv t_5(5n+4)(mod\ 5). \tag{3.17}$$

Agarwal [11,2001] defined the function $P_k(\nu)$ as the number of n- colour partition of ν with subscripts $\leq k$. He then established a bijection between the k- line partitions enumerated by $t_k(\nu)$ and the n- colour partitions enumerated by $P_k(\nu)$. This enabled him to translate the congruences (3.10) - (3.17) into n- colour partition congruences simply by replacing $t_k(\nu)$ by $P_k(\nu)$. In other words the congruences (3.10) - (3.17) can be restated in the following form:

Theorem 3.8. (Agarwal). We have

$$P_2(\nu) \equiv 0(mod\ 5), \quad if \quad \nu \equiv 3 \ or\ 4(mod\ 5) \tag{3.18}$$

$$P_3(3\nu+2) \equiv 0(mod\ 3), \tag{3.19}$$

$$P_2(2\nu) \equiv P_2(2\nu+1)(mod\ 2), \tag{3.20}$$

$$P_3(3\nu) \equiv P_3(3\nu+1)(mod\ 3), \tag{3.21}$$

$$P_4(4\nu) \equiv P_4(4\nu+1) \equiv P_4(4\nu+2)(mod\ 2), \tag{3.22}$$

$$P_4(4\nu+3) \equiv 0(mod\ 2). \tag{3.23}$$

$$P_5(5\nu+1) \equiv P_5(5\nu+3)(mod\ 5), \tag{3.24}$$

$$P_5(5\nu+2) \equiv P_5(5\nu+4)(mod\ 5). \tag{3.25}$$

Analogues to Ramanujan's congruences for $p(n)$, congruences involving coloured and uncoloured F- partitions are also found in the literature. The following two congruences are due to Andrews [26,1984]:

$$\phi_2(5n+3) \equiv c\phi_2(5n+3) \equiv 0(mod\ 5), \tag{3.26}$$

and

$$c\phi_p(n) \equiv 0(mod\ p^2), \tag{3.27}$$

where p is a prime and $p|\ n$.

Kolitsch [42,1985] generalized (3.27) and proved the following congruence

$$\sum_{d|(m,n)} \mu(d)\ c\phi_{\frac{m}{d}}(\frac{n}{d}) \equiv 0(mod\ m^2). \tag{3.28}$$

Obviously, (3.28) reduces to (3.27) when $m = p$, a prime.

The following congruence between coloured and uncoloured F-partitions is found in [36,Garvan,1986]:
For p prime,

$$\phi_{p-1}(n) \equiv c\phi_{p-1}(n)(mod\ p). \tag{3.29}$$

We thus see how Ramanujan's congruences inspired several other mathematicians to pursue research in this area. In fact he himself proved many other identities connected with the congruence properties of partitions, such as

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6}. \tag{3.30}$$

This result was considered to be the representative of the best of Ramanujan's work by G.H. Hardy.

4 Rogers - Ramanujan Identities

The following two "sum- product" identities are known as Rogers - Ramanujan identities:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty}(1-q^{5n-1})^{-1}(1-q^{5n-4})^{-1}, \tag{4.1}$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty}(1-q^{5n-2})^{-1}(1-q^{5n-3})^{-1}. \tag{4.2}$$

They were first discovered by Rogers in 1894 **(L.J. Rogers, Second memoir on the expansion of certain infinite products, Proc. London Math. Soc. 25(1894), 318-343.)** and were rediscovered by Ramanujan in 1913. He had no proof. In 1917 when he was looking through the old volumes of the Proceedings of the London Mathematical Society, came accidentally across Roger's paper. A correspondence followed between Ramanujan and Rogers. Ramanujan published a paper in 1919 **(S. Ramanujan, Proof of certain identities in combinatory analysis, Proceedings of the Cambridge Philosophical Society, XIX, (1919), 214-216)** which contains two proofs (one by Ramanujan and the other by Rogers) and a note by Hardy. After the publication of this paper these identities are known as Rogers - Ramanujan identities. MacMahon [47] gave the following combinatorial interpretations of (4.1) and (4.2), respectively:

Theorem 4.1. The number of partitions of n into parts with the minimal difference 2 equals the number of partitions of n into parts which are congruent to $\pm\ 1(mod\ 5)$.

Theorem 4.2. The number of partitions n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts which are congruent to $\pm\ 2(mod\ 5)$.

The following generalization of Theorems 4.1 and 4.2 is due to Gordon [37]:

Theorem 4.3. (B. Gordon). For $k \geq 2$ and $1 \leq i \leq k$,, let $B_{k,i}(n)$ denote the number of partitions of n of the form $b_1 + b_2 + \cdots + b_s$, where $b_j - b_{j+k-1} \geq 2$, and atmost $i - 1$ of the b_j equal 1. Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm\, i(2k+1)$. Then

$$A_{k,i}(n) = B_{k,i}(n) \quad for \quad all \quad n.$$

Obviously, Theorem 4.1 is the particular case $k = i = 2$ of Theorem 4.3 and Theorem 4.2 is the particular case $k = i + 1 = 2$.

Andrews [24] gave the following analytic counterpart of Theorem 4.3:

Theorem 4.4. (G.E. Andrews). For $1 \leq i \leq k, k, \geq 2, \mid q \mid < 1$.

$$\sum_{n_1,n_2,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\ldots+N_{k-1}^2+N_i+N_{i+1}+\ldots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\ldots(q;q)_{n_{k-1}}}$$
$$= \prod_{n\not\equiv 0,\pm i(mod\ \ 2k+1)} (1-q^n)^{-1}, \tag{4.3}$$

where $N_j = n_j + n_{j+1} + \ldots + n_{k-1.}$

It can be easily seen that Identities (4.1) and (4.2) are the particular cases $k = i = 2$ and $k = i + 1 = 2$ of Theorem 4.4.

Partition theoretic interpretations of many more q- series identities have been given by several mathematicians. See, for instance, Gordon [38], Connor [31], Hirschhorn [40], Agarwal and Andrews [16], Subbarao [57], Subbarao and Agarwal [58]. In all these results only ordinary partitions were used. Analogous to MacMahon's combinatorial interpretations (Theorem 4.1 and

4.2) of the Rogers - Ramanujan identities (4.1) and (4.2), Andrews [25] using n- coloured partitions conjectured and Agarwal [1] proved the following theorems:

Theorem 4.5. The number of partitions with "n copies of n" of ν such that each pair of summands m_i,r_j has positive weighted difference equals the numbers of ordinary partitions of n into parts $\not\equiv 0, \pm 4 (mod\ 10)$.

Example. For $\nu = 6$, we have 8 relevant partitions of each type, viz., 6_1, 6_2, 6_3, 6_4, 6_5, 6_6, $5_1 + 1_1$, $5_2 + 1_1$ of the first type and 51, 3^2, 321, 31^3, 2^3, $2^2 1^2$, 21^4, 1^6 of the second type.

Theorem 4.6. The number of partitions with "n copies of n" of ν such that each pair of summands m_i, r_j has nonnegative weighted difference equals the number of ordinary partitions of n into parts $\not\equiv 0, \pm 6 (mod\ 14)$.

Theorems 4.5 and 4.6 are the combinatorial interpretations of the following q-series identities from [54]:

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q:q)_n (q;q^2)_n} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n})(1-q^{10n-4})(1-q^{10n-6}), (I(46), p.156) \tag{4.4}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q:q)_n (q;q^2)_n} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{14n})(1-q^{14n-6})(1-q^{14n-8}). \ (I(61), p.158) \tag{4.5}$$

The original impetus for Theorems 4.5 and 4.6 came from an attempt to understand the left side of the following identity which arose in Baxter's solution of the hard hexagon model [30, Chap. 14]:

For $\sigma_1 = 0$, $\sigma_i = 0$ or 1, $\sigma_i + \sigma_{i+1} \leq 1$,

$$\sum_{\sigma_2,\cdots,\sigma_m,\cdots} q^{(\sigma_2-\sigma_1\sigma_3).1+(\sigma_3-\sigma_2\sigma_4).2+(\sigma_4-\sigma_3\sigma_5).3+\cdots} = \prod_{n\not\equiv 0,\pm 4 (mod\ 10)} (1-q^n)^{-1}. \tag{4.6}$$

Under the conditions σ_i= 0 or 1 and $\sigma_i+\sigma_{i+1} \leq 1$, each value in the parenthesis is 1 or 0 or -1, so the exponents look like collections of

1
2, $1-2+3$
3, $2-3+4$, $1-2+3-4+5$
4, $3-4+5$, $2-3+4-5+6$, $1-2+3-4+5-6+7$

etc.

We call these collections as blocks and denote them by $1_1, 2_1, 2_2, \cdots$. It was found by Andrews and Baxter that the partitions which we get from the above sum are such that the minimum difference between the blocks is 3.
Now any block m_i is : $(m-i+1) - \cdots \pm (m+i-1)$,
and any block n_j is :$(n-j+1) - \cdots \pm (n+j-1)$.
Now $m_i \geq n_j + 3$ means $m-i+1 \geq n+j-1+3$ or

$$m-n-i-j > 0$$

that is, the weighted difference $((m_i - n_j)) > 0$.

Analogous to Gordon's Theorem (Th. 4.3) which generalizes

MacMahon's combinatorial interpretations of the Rogers- Ramanujan identities, Agarwal and Andrews [17] proved the following generalization of Theorem 4.5 and 4.6:

Theorem 4.7 (Agarwal and Andrews). For $0 \leq t \leq k-1$, $k \geq 2$, let $A_t(k,\nu)$ denote the number of partitions of ν with "$n+t$ copies of n" such that if the weighted difference of any pair of summands m_i, r_j is nonpositive, then it is even and ≥ -2 $\min(i-1, j-1, k-3)$. And if $t \geq 1$, then for some i, i_{i+t} is a part. Let $B_t(k,\nu)$ denote the number of partitions of n into parts $\not\equiv 0, \pm 2(k-t)(mod\ 4k+2)$. Then

$$A_t(k,\nu) = B_t(k,\nu), \quad for \quad all \quad \nu.$$

Obviously, Theorems 4.5 and 4.6 are special cases $k = t+2 = 2$ and $k = t+3 = 3$, respectively, of Theorem 4.7.

As Andrews' Theorem (Th. 4.4) provides an analytic counterpart for Gordon's combinatorial theorem (Th. 4.3), similarly, the following Theorem of Agarwal, Andrews and Bressoud [18] provides an analytic counterpart for Theorem (4.7):

Theorem 4.8 (Agarwal, Andrews and Bressoud). Given $0 \leq t \leq k-1$, $k \geq 2$, let $r = [k/2]$ and $\chi(A)$ is 1 if A is true, 0 otherwise. If $k-r-1 \leq t \leq k-1$, then

$$\sum_{m_1 \geq \ldots \geq m_r \geq 0} \frac{q^{m_1^2+\ldots+m_r^2+a_1m_1+\ldots+a_rm_r+\binom{m_r}{2}\chi(k \text{ is even})}}{(q;q)_{m_1-m_2}\cdots(q;q)_{m_{r-1}-m_r}(q;q)_{m_r}(q;q^2)_{m_r+1}}$$

$$= \prod_{n \not\equiv 0, \pm 2(k-t) (mod \ 4k+2)} (1-q^n)^{-1}, \quad (4.7)$$

where $a_i = 1 + \chi(i \geq k-t)$.

If $0 \leq t \leq r$, then

$$\sum_{m_1 \geq \ldots \geq m_r \geq 0} \frac{q^{m_1^2+\ldots+m_r^2-m_1-\ldots-m_t+\binom{m_r}{2}\chi(k \text{ is even})(1-q^{m_t})}}{(q;q)_{m_1-m_2} \cdots (q;q)_{m_{r-1}-m_r}(q;q)_{m_r}(q;q^2)_{m_r}}$$

$$= \prod_{n \not\equiv 0, \pm 2(k-t) (mod \ 4k+2)} (1-q^n)^{-1}, \quad (4.8)$$

where $q^{-m_1-\ldots-m_t}(1-q^{m_t})$ is defined to be one if t is zero.

Identities (4.4) and (4.5) are special cases $k = t+2 = 2$ and $k = t+3 = 3$, respectively of Theorem 4.8.

For more Rogers- Ramanujan type identities involving coloured partitions, lattice paths and the Frobenius partitions the reader is referred to
Agarwal [2,3,4,5,6,7,8,9,12,15]. The Rogers- Ramanujan Identities are considered among the most beautiful formulae in Mathematics. Their study continues to be a very active area of research even today. They have applications in different areas. G. Andrews and R. Askey have demonstrated a very intimate connection between the Rogers- Ramanujan identities and certain families of orthogonal polynomails. J. Lepowsky, A. Feingold, S. Milne and R. Wilson have found a connection with Lie alge-

bras. F. Dyson has found applications of Rogers- Ramanujan identities in particle physics and R.J. Baxter in statistical mechanism.

5 Concluding Remarks

We have only touched Ramanujan's contribution to theory of partitions which represent only a small part of his contributions to mathematics. But they are more than enough to prove the profoundness and invincible originality of his work. One of the most remarkable features of his work is: it remains youthful in this modern world of computers even after more than eight decades of his death. We have seen in the preceding sections how his exact formula for $p(n)$, his congruences and identities have inspired numerous mathematicians around the globe and continue to excite researchers today. We hope that they will continue to inspire generations to come.

References

1. A.K. Agarwal, Partitions with "N copies of N", Combinatorie enumerative (Montreal, Que., 1985), Lecture Notes in Math., 1234, Springer, Berlin, (1986), 1-4.

2. A.K. Agarwal, Rogers- Ramanujan identities for n-color partitions, J. Number Theory 28 (1988), no. 3, 299-306.

3. A.K. Agarwal, New combinatorial interpretations of two analytic identities, Proc. Amer. Math. Soc. 107(1989) no. 2, 561-567.

4. A.K. Agarwal, Bijective proofs of some n- color partition identities, Canad. Math. Bull. 32(1989),no.3, 327-332.

5. A.K. Agarwal, Antihook differences and some partition identities, Proc. Amer. Math. Soc. 116(1990), no. 4, 1137-1142.

6. A.K. Agarwal, q-functional equations and some partition identities, combinatorics and theoretical computer science (Washington, DC, 1989), Discrete Appl. Math. 34(1991), no. 1-3, 17-26.

7. A.K. Agarwal, New classes of infinite 3-way partition identities, Ars Combin. 44(1996), 33-54.

8. A.K. Agarwal, Lattice paths and n-color partitions, Util. Math. 53(1998), 71-80.

9. A.K. Agarwal, Identities and generating functions for certain classes of F- partitions, Ars Combin. 57(2000), 65-75.

10. A.K. Agarwal, n-colour compositions, Indian J. Pure Appl. Math. 31(2000), no. 11, 1421-27.

11. A.K. Agarwal, Ramanujan congruences for n-colour partitions, Math. Student 70(2001), no. 1-4, 199-204.

12. A.K. Agarwal, An analogue of Euler's identity and new combinatorial properties of n-colour compositions, J. Comput. Appl. Math. 160 (2003), no. 1-2, 9-15.

13. A.K. Agarwal, An extension of Euler's theorem, J. Indian Math. Soc. (N.S.), 70(2003), no. 1-4, 17-24.

14. A.K. Agarwal, n-color partition theoretic interpretations of some mock theta functions, Electron. J. Combin. 11(2004), no.1, Note 14, 6pp
(electronic).

15. A.K. Agarwal, New classes of combinatorial identities, Ars Combin. 76(2005), 151-160.

16. A.K. Agarwal and G.E. Andrews, Hook differences and lattice paths, J. statist. Plann. inference 14(1986), no.1, 5-14.

17. A.K. Agarwal and G.E. Andrews, Rogers-Ramanujan identities for partitions with "N copies of N" , J. Combin. Theory Ser A 45(1987), no.1, 40-49.

18. A.K. Agarwal, G.E. Andrews and D.M. Bressoud, The Bailey lattice, J. Indian Math. Soc. (N.S.) 51(1987), 57-73 (1988).

19. A.K. Agarwal and R. Balasubramanian, n- color partitions with weighted differences equal to minus two, Internat. J. Math. Sci. 20(1997), no.4, 759-768.

20. A.K. Agarwal and D.M. Bressoud, Lattice paths and multiple basic hypergeometric series, Pacific J. Math. 136(1989), no.2, 209-228.

21. A.K. Agarwal and M. Rana, Two new combinatorial interpretations of a fifth order mock theta function, J. Indian Math.Soc. (N.S.), to appear.

22. A.K. Agarwal and M.V.Subbarao, Some Properties of perfect partitions, Indian J. Pure Appl. Math. 22(1991), no. 9, 737-743.

23. S. Anand and A.K. Agarwal, On some restricted plane partition functions, Util. Math. 74 (2007), 121-130.

24. G.E. Andrews, Interactions of additive number theory and statistical mechanics, Abstracts Amer. Math. Soc. 3(1982), 3.

25. G.E. Andrews, Generalized Frobenius Partitions, Mem. Amer. Math. Soc. 49(1984), No. 301, IV +44pp.

26. G.E. Andrews and F.G. Garvan, Dyson's crank of a partition, Bulletin (New Series) Amer. Math. Soc. 18(2), (1988), 167-171.

27. A.O.L. Atkin, Proof of a conjecture of Ramanujan, Glasgow Math. J. 8(1967), 14-32.

28. A.O.L. Atkin and H.P.F. Swinnerton - Dyer, Some properties of partitions, Proc. London Math. Soc. (3) 4(1953), 84-106.

29. R.J. Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press, London, 1982.

30. W.G. Connor, Partitions theorems related to some identities of Rogers and Watson, Trans. Amer. Math. Soc. 214(1975), 95-111.

31. M.S. Cheema and B.Gordon, Some remarks on two - and three - line partitions, Duke Math. J., 31(1964), 267-273.

32. S. Chowla, Congruence properties of partitions, J. London Math. Soc. 9(1934), 147-153.

33. F.J. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8(1944), 10-15.

34. J.M. Gandhi, Some congruences for k-line partitions of a number, Amer. Math. Monthly 74(1967), 179-181.

35. F.G. Garvan, Generalizations of Dyson's rank, Ph.D. thesis, Pennsylvania State University (1986), VIII +127pp.

36. B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities, Amer. J. Math. 83(1961), 393-399.

37. B. Gordon, Some Continued Fractions of the Rogers- Ramanujan type, Duke Math. J. 32(1965), 741-748.

38. G.H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, Proc. London Math. Soc. (2) 17(1918), 75-115.

39. M.D. Hirschhorm, Some partition theorems of the Rogers-Ramanujan type, J. Combin. theory Ser A, 27(1) (1979), 33-37.

40. W. Krečmar, Surles properties de la divisibilite d' une fonction additive, Bull. Acad. Sci, URSS (1933), 763-800.

41. L.W. Kolitsch, Some analytic and arithmetic properties of generalized Frobenius Partitions, Ph.D. Thesis, Pennsylvania State University, 1985.

42. D.H. Lehmer, On a conjecture of Ramanujan, J. London Math. Soc., 11(1936), 114-118.

43. D.H. Lehmer, On the remainders and convergence of the series for the partition function, Trans. Amer. Math. Soc. (46) (1939), 363-373.

44. J. Lehner, Proof of Ramanujan's partition congruence for the modulus 11^3, Proc. Amer. Math. Soc. 1 (1950), 172-181.

45. W.J. LeVeque, Reviews in Number Theory, 4, Amer. Math. Soc., Providence, R.I. (1974).

46. P.A. MacMahon, Combinatory Analysis, Vol.2, Cambridge Univ. Press, London and New York (1916).

47. G. Narang and A.K. Agarwal, n-colour self- inverse compositions, proc. Indian Acad. Sci. (math. Sci.), 116(3), (2006), 257-266.

48. G. Narang and A.K. Agarwal, Lattice paths and n-colour compositions, Discrete Mathematics, 308(2008), 1732-1740.

49. H. Rademacher, On a partition function $p(n)$, Proc. London Math. Soc. (2) 43, (1937), 241-254.

50. H. Rademacher, On the expansion of the partition function in a series, Ann. of Math., 44(1943), 416-422.

51. H. Rademacher, Topics in Analytic Number Theory, (1973), Springer, Berlin.

52. S. Ramanujan, Some properties of $p(n)$, the number of partitions of n, Proc. Camb. Phil. Soc. XIX (1919), 207-210.

53. L.J. Slater, Further identities of the Rogers-Ramanujan type, Proc. London Math. Soc. Ser.2, 54(1952), 147-167.

54. R.P. Stanley, Theory and applications of plane partitions, I, Stud. Appl. Math. 50(1971), 167-188.

55. R.P. Stanley, Theory and applications of plane partitions, II, Stud. Appl. Math. 50(1971), 259-279.

56. M.V. Subbarao, Some Rogers- Ramanujan type partition theorems, Pacific J. Math. 120(1985), 431-435.

57. M.V. Subbarao and A.K. Agarwal, Further theorems of the Rogers- Ramanujan type, Canad. Math. Bull. 31(2), (1988), 210-214.

58. G.N. Watson, Ramanujan Vermutung uber Zerfallunsan zahlen, J. reine angew. Math. 179(1936), 97-128.

59. L. Winquist, An elementary proof of $p(11m+6) \equiv 0(mod\, 11)$, J. Combin. Theory, 6(1969), 456-59.